ESTIMATING THE STRESS-STRENGTH PARAMETER OF LOG-UNIFORM DISTRIBUTIONS AND ITS APPLICATIONS IN EDUCATIONAL PSYCHOLOGY AND ELECTRICAL ENGINEERING

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Abstract. *In this paper, we develop and analyze estimators of the stress-strength parameter for log-uniform distributions, with applications in both educational psychology and electrical engineering. Specifically, we propose a shrinkage estimator of the stress-strength parameter, constructed based on its maximum likelihood estimator and uniformly minimum variance unbiased estimator. A comprehensive simulation study is conducted to assess the behavior of these estimators. As practical applications, we investigate the effect of coffee consumption on students performance in statistics exams and analyze the degradation behavior of lithium-ion batteries under different thermal storage conditions by estimating the stress-strength parameter from real datasets. The results demonstrate the effectiveness of the proposed estimators and provide meaningful insights into these diverse fields.*

Keywords: *Stress-strength parameter, Log-uniform distribution, Shrinkage estimator, Educational psychology, Electrical engineering.*

1. INTRODUCTION

In todays world, the concept of stress carries a broad range of meanings across different disciplines. In psychology, stress refers to any stimulus that induces physical, emotional, or cognitive strain. In engineering—particularly in electrical engineering—stress is commonly used to describe external conditions that

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challenge the performance and stability of technical systems. Despite these disciplinary differences, both domains share a central theme: the dynamic interaction between stress and strength. In psychological contexts, this interaction can be observed in how individuals respond to pressure, such as students managing exam-related stress. For instance, one line of research investigates the impact of caffeinated beverages like coffee on students performance in statistics exams-a topic situated within educational psychology. In engineering, a parallel can be drawn in the analysis of lithium-ion batteries, where thermal stress during storage influences the degradation and long-term reliability of energy systems. These two seemingly distinct applications illustrate the universal relevance of stress-strength models across human and technological systems.

One statistical framework that explicitly models the interaction between stress and strength is the stress-strength model, originally introduced by Birnbaum (1956). In this model, stress and strength are represented as independent random variables, typically denoted by Y and X, respectively. The probability that the strength exceeds the stress, i.e., $P(X > Y)$, is referred to as the stress-strength parameter, denoted by R. This model has found widespread application in various fields, including psychology, medicine, education, and engineering. A comprehensive summary of key theoretical developments and applications of the stress-strength model can be found in Kotz et al. (2003).

Since both stress and strength are treated as random variables in the stress-strength model, numerous studies have focused on estimating the parameter $R = P(X > Y)$ under various distributional assumptions for X and Y. Recent contributions in this area include the works of Chacko and Mathew (2019), Akgul et al. (2021), Kotb and Raqab (2021), Rezaei and Yousefzadeh (2022), and Erbayram et al. (2025).

The primary aim of this study is twofold. From the perspective of educational psychology, we examine the effect of coffee consumption-as a source of caffeine-on students performance in statistics exams, using the stress-strength framework to model the relationship between performance (strength) and exam-related stress. From the standpoint of electrical engineering, we employ the same framework to investigate the degradation behavior of lithium-ion batteries stored under varying thermal conditions, where thermal stress acts against the battery's intrinsic strength and reliability over time.

In both applications, we use the log-uniform (reciprocal) distribution to model the stress and strength variables. This distribution was first introduced by Hamming (1970) and is a continuous, right-skewed distribution. It is especially useful in situations where the variables span a wide range or when there is limited information about their true scale. Because of these properties, the log-uniform distribution has been widely used as a non-informative prior for scale parameters in Bayesian analysis. Its simplicity and flexibility also make it a practical choice for modeling phenomena with multiplicative variability, such as stress in physical systems or uncertainty in behavioral responses. Recent studies exploring the properties and applications of this distribution include Escobar and Moreno-Jiménez (2000), Nguyen et al. (2004), Kreinovich et al. (2015), Shaji and Sebastian (2023), and Av and Sebastian (2025).

The probability density function (pdf) and the cumulative distribution function (cdf) of the log-uniform distribution with parameter $b > 1$ are given by

$$f_Z(z;b) = \frac{1}{z \ln b}, \quad \frac{1}{b} \leq z \leq 1, \tag{1}$$

and

$$F_Z(z;b) = 1 + \frac{\ln z}{\ln b}, \quad \frac{1}{b} \leq z \leq 1, \tag{2}$$

respectively. A random variable Z is said to follow a log-uniform distribution with parameter b, denoted by $Z \sim LU(b)$, if its pdf is given by (1).

This paper is organized as follows: In Section 2, we derive the stress-strength parameter for log-uniform distributions. Sections 3 and 4 present the maximum likelihood estimator and the uniformly minimum variance unbiased estimator of the stress-strength parameter, respectively. Based on these estimators, a shrinkage estimator is proposed in Section 5. Section 6 provides a simulation study to assess the performance of the proposed estimators. Finally, Section 7 demonstrates the practical application of the proposed methods by investigating the effect of coffee consumption on students performance in statistics exams within educational psychology, as well as analyzing the degradation behavior of lithium-ion batteries under different thermal storage conditions in electrical engineering.

2. STRESS-STRENGTH PARAMETER

In this section, we compute the stress-strength parameter for the log-uniform distributions. Suppose that $X \sim LU(b_1)$ and $Y \sim LU(b_2)$ are two independent random variables, where b_1 and b_2 are unknown parameters. The probability $R = P(X > Y)$ can be expressed as

$$R = \int_{\frac{1}{b_1}}^{1} F_Y(x;b_2) f_X(x;b_1)\, dx,$$

where f_X and F_Y denote the pdf of X and the cdf of Y, respectively.

Since $f_X(x;b_1)$ is nonzero only for $x \in \left[\frac{1}{b_1}, 1\right]$, and $F_Y(x;b_2) = 0$ for $x < \frac{1}{b_2}$, the integration limits depend on the relation between $\frac{1}{b_1}$ and $\frac{1}{b_2}$:

- If $\frac{1}{b_1} \geq \frac{1}{b_2}$, the entire interval $\left[\frac{1}{b_1}, 1\right]$ lies within the support of F_Y, so the integral starts at $\frac{1}{b_1}$.
- If $\frac{1}{b_1} < \frac{1}{b_2}$, the integration begins at $\frac{1}{b_2}$ because $F_Y(x;b_2) = 0$ for $x \in \left[\frac{1}{b_1}, \frac{1}{b_2}\right)$.

Defining $a^* = \max\left\{\frac{1}{b_1}, \frac{1}{b_2}\right\}$, we write

$$R = P(X > Y) = \int_{a^*}^{1} F_Y(x;b_2) f_X(x;b_1) dx.$$

For $x \in [a^*, 1] \subseteq \left[\frac{1}{b_2}, 1\right]$, we have

$$f_X(x;b_1) = \frac{1}{x \ln b_1} \quad \text{and} \quad F_Y(x;b_2) = 1 + \frac{\ln x}{\ln b_2}.$$

Therefore,

$$R = \int_{a^*}^{1} \left(1 + \frac{\ln x}{\ln b_2}\right) \frac{1}{x \ln b_1} dx = \frac{1}{\ln b_1} \int_{a^*}^{1} \frac{1}{x} dx + \frac{1}{\ln b_1 \ln b_2} \int_{a^*}^{1} \frac{\ln x}{x} dx.$$

We know that $\int_{a^*}^{1} \frac{1}{x} dx = -\ln a^*$. Using integration by parts,

$$\int_{a^*}^{1} \frac{\ln x}{x} dx = \frac{1}{2} (\ln x)^2 \Big|_{a^*}^{1} = -\frac{1}{2} (\ln a^*)^2.$$

Hence,

$$R = \frac{-\ln a^*}{\ln b_1} - \frac{(\ln a^*)^2}{2 \ln b_1 \ln b_2}, \quad b_1 > 1, \quad b_2 > 1.$$

Substituting back for a^*, the final expression for R is

$$R = \begin{cases} \frac{\ln b_2}{2 \ln b_1} & \text{if } \frac{1}{b_1} < \frac{1}{b_2}, \\ 1 - \frac{\ln b_1}{2 \ln b_2} & \text{if } \frac{1}{b_1} \geq \frac{1}{b_2}. \end{cases} \tag{3}$$

As derived in (3), the stress-strength parameter R is a function of the unknown parameters b_1 and b_2, and is denoted by $R = R(b_1, b_2)$. Depending on the values of b_1 and b_2, the expression and range of R vary as follows:

- If $\frac{1}{b_1} < \frac{1}{b_2}$, then $R = \frac{\ln b_2}{2 \ln b_1}$, which lies in the interval $\left(0, \frac{1}{2}\right)$;
- If $\frac{1}{b_1} \geq \frac{1}{b_2}$, then $R = 1 - \frac{\ln b_1}{2 \ln b_2}$, which lies in the interval $\left[\frac{1}{2}, 1\right)$.

In the following three sections, different estimators of R are developed using the methods of maximum likelihood estimation (MLE), uniformly minimum variance unbiased estimation (UMVUE), and shrinkage estimation. To this end, let $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ be two independent random samples drawn from the log-uniform distributions $LU(b_1)$ and $LU(b_2)$, respectively.

3. MLE of R

In this section, the MLE of R is derived. Since the stress-strength parameter R is a function of the parameters b_1 and b_2, it is necessary to first obtain their MLEs in order to apply the invariance property of MLEs.

Let $Z_1, Z_2, \dots, Z_N$ be a random sample from the distribution $LU(b)$. Based on the pdf in (1), the likelihood function is given by

$$\begin{aligned} L(b) &= \prod_{i=1}^{N} f(z_i; b) = \prod_{i=1}^{N} \frac{1}{z_i \ln b} I_{\left[\frac{1}{b}, 1\right]}(z_i) \\ &= \left(\frac{1}{\ln b}\right)^N \left(\prod_{i=1}^{N} \frac{1}{z_i}\right) I_{\left[\frac{1}{z_{(1)}}, +\infty\right]}(b) \, I_{[-\infty, 1]}(z_{(N)}), \end{aligned}$$

where $z_{(j)}$ denotes the jth order statistic. Since $L(b)$ is a decreasing function of b, the MLE of b is $\hat{b} = \frac{1}{z_{(1)}}$.

Therefore, based on two independent samples $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ from $LU(b_1)$ and $LU(b_2)$ respectively, their MLEs are $\hat{b}_1 = \frac{1}{X_{(1)}}$ and $\hat{b}_2 = \frac{1}{Y_{(1)}}$. Consequently, the MLE of the stress-strength parameter R is

$$\hat{R}_{MLE} = \begin{cases} \frac{\ln Y_{(1)}}{2 \ln X_{(1)}} & \text{if } X_{(1)} < Y_{(1)}, \\ 1 - \frac{\ln X_{(1)}}{2 \ln Y_{(1)}} & \text{if } X_{(1)} \geq Y_{(1)}. \end{cases} \tag{4}$$

To further investigate the statistical properties of $\hat{R}_{MLE}$, its pdf is presented below.

Theorem 3.1. *The pdf of $\hat{R}_{MLE}$ is given by*

$$f_{\hat{R}_{MLE}}(r) = \begin{cases} \dfrac{2^m nm}{n+m}\left(\dfrac{\ln b_1}{\ln b_2}\right)^m r^{m-1} & \text{if } 0 \le r \le \frac{\ln b_2}{2\ln b_1}, \\ \dfrac{2^n nm}{n+m}\left(\dfrac{\ln b_2}{\ln b_1}\right)^n (1-r)^{n-1} & \text{if } 1-\frac{\ln b_1}{2\ln b_2} \le r \le 1. \end{cases}$$

Proof. Consider the case $X_{(1)} \ge Y_{(1)}$. From (4), we have the transformation

$$x = \exp\left\{2(1-r)\ln y\right\}, \quad |J| = \left|\frac{\partial x}{\partial r}\right| = -2\ln y \cdot \exp\left\{2(1-r)\ln y\right\}. \tag{5}$$

Thus, for $0 \le r \le 1$ and $\frac{1}{b_2} \le y < 1$, the joint density becomes

$$\begin{aligned} f_{\hat{R}_{MLE},Y_{(1)}}(r,y) &= |J| \cdot f_{X_{(1)},Y_{(1)}}\big(x(r,y),y\big) \\ &= 2^n nm\left(\frac{1}{\ln b_1}\right)^n\left(\frac{1}{\ln b_2}\right)^m (-1)^{n+m-1}(1-r)^{n-1}\frac{(\ln y)^{n+m-1}}{y}. \end{aligned}$$

Here,

$$f_{X_{(1)},Y_{(1)}}(x,y) = \frac{nm}{xy}\left(\frac{1}{\ln b_1}\right)^n\left(\frac{1}{\ln b_2}\right)^m\left(\ln\frac{1}{x}\right)^{n-1}\left(\ln\frac{1}{y}\right)^{m-1}, \tag{6}$$

for $\frac{1}{b_1} \le x \le 1$ and $\frac{1}{b_2} \le y \le 1$. Integrating out y gives

$$\begin{aligned} f_{\hat{R}_{MLE}}(r) &= \int_{\frac{1}{b_2}}^{1} f_{\hat{R}_{MLE},Y_{(1)}}(r,y)dy \\ &= 2^n nm\left(\frac{1}{\ln b_1}\right)^n\left(\frac{1}{\ln b_2}\right)^m (-1)^{n+m-1}(1-r)^{n-1}\int_{\frac{1}{b_2}}^{1}\frac{(\ln y)^{n+m-1}}{y}dy \\ &= \frac{2^n nm}{n+m}\left(\frac{\ln b_2}{\ln b_1}\right)^n (1-r)^{n-1}, \end{aligned}$$

since the integral evaluates to $\frac{1}{n+m}(\ln b_2)^{n+m}$.

From the change of variable in (5), we have $x = \exp\{2(1-r)\ln y\}$, which must satisfy $\frac{1}{b_1} \le x \le 1$. This leads to the condition $\left(\frac{1}{b_1}\right)^{\frac{1}{2(1-r)}} \le y \le 1$. Since $y \in \left[\frac{1}{b_2}, 1\right]$, the inequality $\left(\frac{1}{b_1}\right)^{\frac{1}{2(1-r)}} \le \frac{1}{b_2}$ must be satisfied. Taking logarithms on both sides results in the constraint

$$r \ge 1 - \frac{\ln b_1}{2\ln b_2},$$

which determines the lower bound of the support of $f_{\hat{R}_{MLE}}(r)$. Therefore, the support of $f_{\hat{R}_{MLE}}(r)$ is $\left[1-\frac{\ln b_1}{2\ln b_2},1\right]$ in this case.

Similarly, for the case $X_{(1)} < Y_{(1)}$, using the transformation $y=\exp\{2r\ln x\}$, it can be shown that

$$f_{\hat{R}_{MLE}}(r) = \frac{2^m nm}{n+m}\left(\frac{\ln b_1}{\ln b_2}\right)^m r^{m-1}, \quad \text{for } 0 \le r \le \frac{\ln b_2}{2\ln b_1}.$$

□

The first and second moments of $\hat{R}_{MLE}$ are derived in the following corollary.

Corollary 3.1. *Consider the MLE of R in* (4), *then*

$$\begin{aligned} E(\hat{R}_{MLE}) =& \frac{nm}{2(n+m)(m+1)}\frac{\ln b_2}{\ln b_1} + \frac{m}{n+m} - \frac{nm}{2(n+m)(n+1)}\frac{\ln b_1}{\ln b_2}, \\ E(\hat{R}^2_{MLE}) =& \frac{nm}{4(n+m)(m+2)}\left(\frac{\ln b_2}{\ln b_1}\right)^2 + \frac{m}{n+m} - \frac{nm}{(n+m)(n+1)}\frac{\ln b_1}{\ln b_2} \\ &+ \frac{nm}{4(n+m)(n+2)}\left(\frac{\ln b_1}{\ln b_2}\right)^2. \end{aligned}$$

Proof. Using the pdf of $\hat{R}_{MLE}$ given in Theorem 3.1, the expectation is computed as

$$\begin{aligned} E(\hat{R}_{MLE}) &= \int_0^1 r f_{\hat{R}_{MLE}}(r)\,dr = \int_0^{\frac{\ln b_2}{2\ln b_1}} r f_{\hat{R}_{MLE}}(r)\,dr + \int_{1-\frac{\ln b_1}{2\ln b_2}}^1 r f_{\hat{R}_{MLE}}(r)\,dr \\ &= \frac{2^m nm}{n+m}\left(\frac{\ln b_1}{\ln b_2}\right)^m \int_0^{\frac{\ln b_2}{2\ln b_1}} r^m\,dr + \frac{2^n nm}{n+m}\left(\frac{\ln b_2}{\ln b_1}\right)^n \int_{1-\frac{\ln b_1}{2\ln b_2}}^1 r(1-r)^{n-1}\,dr. \end{aligned}$$

Similarly, the second moment is obtained as

$$\begin{aligned} E(\hat{R}^2_{MLE}) &= \int_0^1 r^2 f_{\hat{R}_{MLE}}(r)\,dr = \int_0^{\frac{\ln b_2}{2\ln b_1}} r^2 f_{\hat{R}_{MLE}}(r)\,dr + \int_{1-\frac{\ln b_1}{2\ln b_2}}^1 r^2 f_{\hat{R}_{MLE}}(r)\,dr \\ &= \frac{2^m nm}{n+m}\left(\frac{\ln b_1}{\ln b_2}\right)^m \int_0^{\frac{\ln b_2}{2\ln b_1}} r^{m+1}\,dr + \frac{2^n nm}{n+m}\left(\frac{\ln b_2}{\ln b_1}\right)^n \int_{1-\frac{\ln b_1}{2\ln b_2}}^1 r^2(1-r)^{n-1}\,dr. \end{aligned}$$

Now, evaluating the integrals yields:

$$\int_0^{\frac{\ln b_2}{2\ln b_1}} r^m \, dr = \frac{1}{2^{m+1}(m+1)} \left(\frac{\ln b_2}{\ln b_1}\right)^{m+1},$$

$$\int_0^{\frac{\ln b_2}{2\ln b_1}} r^{m+1} \, dr = \frac{1}{2^{m+2}(m+2)} \left(\frac{\ln b_2}{\ln b_1}\right)^{m+2},$$

$$\int_{1-\frac{\ln b_1}{2\ln b_2}}^{1} r(1-r)^{n-1} \, dr = \frac{1}{2^n} \left(\frac{\ln b_1}{\ln b_2}\right)^{n} \left(\frac{1}{n} - \frac{1}{2(n+1)} \frac{\ln b_1}{\ln b_2}\right),$$

$$\int_{1-\frac{\ln b_1}{2\ln b_2}}^{1} r^2(1-r)^{n-1} \, dr = \frac{1}{2^n} \left(\frac{\ln b_1}{\ln b_2}\right)^{n} \left(\frac{1}{n} - \frac{1}{n+1} \frac{\ln b_1}{\ln b_2} + \frac{1}{4(n+2)} \left(\frac{\ln b_1}{\ln b_2}\right)^2\right).$$

Substituting these expressions into the formulas for $E(\hat{R}_{MLE})$ and $E(\hat{R}^2_{MLE})$ completes the proof. □

4. UMVUE of R

This section is concerned with obtaining the UMVUE of R. The following lemma plays a key role in deriving the UMVUE of R.

Lemma 4.1. *Let $Z_1, Z_2, \ldots, Z_N$ be a random sample from the log-uniform distribution $LU(b)$. Then, the minimum order statistic $T(\boldsymbol{Z}) = Z_{(1)}$ is a complete sufficient statistic for the parameter b.*

Proof. Consider the functions

$$g(z_{(1)}, b) = \left(\frac{1}{\ln b}\right)^N I_{\left[\frac{1}{b}, +\infty\right)}(z_{(1)}), \quad h(\boldsymbol{z}) = \left(\prod_{i=1}^{N} \frac{1}{z_i}\right) I_{(-\infty,1]}(z_{(N)}).$$

Then, the joint pdf can be factorized as

$$f(\boldsymbol{z}; b) = \prod_{i=1}^{N} f(z_i; b) = g(z_{(1)}, b)\, h(\boldsymbol{z}).$$

By the factorization theorem, $T(\boldsymbol{Z}) = Z_{(1)}$ is a sufficient statistic for b, whose pdf is

$$f_T(t) = \frac{N}{t} \left(\frac{1}{\ln b}\right)^N \left(\ln \frac{1}{t}\right)^{N-1}, \quad \frac{1}{b} \le t \le 1.$$

To prove completeness, let $h(t)$ be a function such that

$$E[h(T)] = N \left(\frac{1}{\ln b}\right)^N \int_{\frac{1}{b}}^{1} \frac{h(t)}{t} \left(\ln \frac{1}{t}\right)^{N-1} dt = 0, \quad \forall b > 1.$$

Differentiating with respect to b, we get

$$\frac{\partial}{\partial b}\int_{\frac{1}{b}}^{1}\frac{h(t)}{t}\left(\ln\frac{1}{t}\right)^{N-1}dt=\frac{h\left(\frac{1}{b}\right)}{b}(\ln b)^{N-1}=0,\quad \forall b>1.$$

This implies $h\left(\frac{1}{b}\right)=0$ for all $b>1$, so $h(t)=0$ for $\frac{1}{b}\leq t<1$. Hence, $T(\boldsymbol{Z})=Z_{(1)}$ is a complete sufficient statistic for b. □

The UMVUE of the stress-strength parameter R is derived as follows.

Theorem 4.1. *The UMVUE of R is given by*

$$\hat{R}_{UMVUE}=\begin{cases}\dfrac{(n-1)(m+1)}{2nm}\dfrac{\ln Y_{(1)}}{\ln X_{(1)}} & \text{if } X_{(1)}<Y_{(1)},\\ 1-\dfrac{(n+1)(m-1)}{2nm}\dfrac{\ln X_{(1)}}{\ln Y_{(1)}} & \text{if } X_{(1)}\geq Y_{(1)}.\end{cases}$$

Proof. By Lemma 4.1, $X_{(1)}$ and $Y_{(1)}$ are complete sufficient statistics for b_1 and b_2, respectively. Let $h(X_{(1)},Y_{(1)})$ is an unbiased estimator of $R=P(X>Y)$, i.e.,

$$R=E[h(X_{(1)},Y_{(1)})]=\int_{\frac{1}{b_1}}^{1}\int_{\frac{1}{b_2}}^{1}h(x,y)\,f_{X_{(1)},Y_{(1)}}(x,y)\,dx\,dy,$$

where $f_{X_{(1)},Y_{(1)}}$ is the joint density function (6). Recall that

$$R=\begin{cases}\frac{\ln b_2}{2\ln b_1} & \text{if } \frac{1}{b_1}<\frac{1}{b_2},\\ 1-\frac{\ln b_1}{2\ln b_2} & \text{if } \frac{1}{b_1}\geq\frac{1}{b_2},\end{cases}$$

which can be rewritten as

$$R=\frac{\ln b_2}{2\ln b_1}I\left(\frac{1}{b_1}<\frac{1}{b_2}\right)+\left(1-\frac{\ln b_1}{2\ln b_2}\right)I\left(\frac{1}{b_1}\geq\frac{1}{b_2}\right),$$

where $I(\cdot)$ is the indicator function. Hence,

$$\begin{aligned}&\frac{\ln b_2}{2\ln b_1}I\Big(\frac{1}{b_1}<\frac{1}{b_2}\Big)+\big(1-\frac{\ln b_1}{2\ln b_2}\big)I\Big(\frac{1}{b_1}\geq\frac{1}{b_2}\Big)\\ &=nm\Big(\frac{1}{\ln b_1}\Big)^{n}\Big(\frac{1}{\ln b_2}\Big)^{m}\int_{\frac{1}{b_1}}^{1}\int_{\frac{1}{b_2}}^{1}h(x,y)\frac{1}{xy}\Big(\ln\frac{1}{x}\Big)^{n-1}\Big(\ln\frac{1}{y}\Big)^{m-1}dy\,dx.\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}(\ln b_1)^{n-1}(\ln b_2)^{m+1} I\Big(\frac{1}{b_1} < \frac{1}{b_2}\Big) + (\ln b_1)^n (\ln b_2)^m I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
&\quad - \frac{1}{2}(\ln b_1)^{n+1}(\ln b_2)^{m-1} I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
&= nm \int_{\frac{1}{b_1}}^{1} \int_{\frac{1}{b_2}}^{1} h(x,y) \frac{1}{xy} \Big(\ln \frac{1}{x}\Big)^{n-1} \Big(\ln \frac{1}{y}\Big)^{m-1} dy\, dx.
\end{aligned}$$

By using the Leibniz integral rule, the derivative of the above equation with respect to b_1 is

$$\begin{aligned}
&\frac{n-1}{2b_1}(\ln b_1)^{n-2}(\ln b_2)^{m+1} I\Big(\frac{1}{b_1} < \frac{1}{b_2}\Big) + \frac{n}{b_1}(\ln b_1)^{n-1} (\ln b_2)^m I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
&\quad - (\frac{n+1}{2b_1})(\ln b_1)^{n}(\ln b_2)^{m-1} I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
&= \frac{nm}{b_1}(\ln b_1)^{n-1} \int_{\frac{1}{b_2}}^{1} h(\frac{1}{b_1}, y) \frac{1}{y} \Big(\ln \frac{1}{y}\Big)^{m-1} dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{n-1}{2}(\ln b_1)^{-1}(\ln b_2)^{m+1} I\Big(\frac{1}{b_1} < \frac{1}{b_2}\Big) + n(\ln b_2)^m I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
&\quad - (\frac{n+1}{2})\ln b_1 (\ln b_2)^{m-1} I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
&= nm \int_{\frac{1}{b_2}}^{1} h(\frac{1}{b_1}, y) \frac{1}{y} \Big(\ln \frac{1}{y}\Big)^{m-1} dy.
\end{aligned}$$

Again by using the Leibniz integral rule we have

$$\begin{aligned}
&\frac{(n-1)(m+1)}{2b_2}(\ln b_1)^{-1}(\ln b_2)^{m} I\Big(\frac{1}{b_1} < \frac{1}{b_2}\Big) + \frac{nm}{b_2}(\ln b_2)^{m-1} I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
&\quad - \frac{(n+1)(m-1)}{2b_2}(\ln b_1)(\ln b_2)^{m-2} I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
&= \frac{nm}{b_2}(\ln b_2)^{m-1} h\Big(\frac{1}{b_1}, \frac{1}{b_2}\Big).
\end{aligned}$$

So,

$$
\begin{aligned}
h\Big(\frac{1}{b_1},\frac{1}{b_2}\Big) =& \frac{(n-1)(m+1)}{2nm}\frac{\ln b_2}{\ln b_1} I\Big(\frac{1}{b_1} < \frac{1}{b_2}\Big) \\
& + \Big(1 - \frac{(n+1)(m-1)}{2nm}\frac{\ln b_1}{\ln b_2}\Big) I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big) \\
=& \frac{(n-1)(m+1)}{2nm}\frac{\ln \frac{1}{b_2}}{\ln \frac{1}{b_1}} I\Big(\frac{1}{b_1} < \frac{1}{b_2}\Big) \\
& + \Big(1 - \frac{(n+1)(m-1)}{2nm}\frac{\ln \frac{1}{b_1}}{\ln \frac{1}{b_2}}\Big) I\Big(\frac{1}{b_1} \geq \frac{1}{b_2}\Big),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\hat{R}_{UMVUE} =& h(X_{(1)}, Y_{(1)}) \\
=& \frac{(n-1)(m+1)}{2nm}\frac{\ln Y_{(1)}}{\ln X_{(1)}} I\Big(X_{(1)} < Y_{(1)}\Big) \\
& + \Big(1 - \frac{(n+1)(m-1)}{2nm}\frac{\ln X_{(1)}}{\ln Y_{(1)}}\Big) I\Big(X_{(1)} \geq Y_{(1)}\Big), \\
=& \begin{cases} \frac{(n-1)(m+1)}{2nm}\frac{\ln Y_{(1)}}{\ln X_{(1)}} & \text{if} \quad X_{(1)} < Y_{(1)}, \\ 1 - \frac{(n+1)(m-1)}{2nm}\frac{\ln X_{(1)}}{\ln Y_{(1)}} & \text{if} \quad X_{(1)} \geq Y_{(1)}. \end{cases}
\end{aligned}
$$

□

The following theorem shows that $\hat{R}_{\text{UMVUE}}$ and $\hat{R}_{\text{MLE}}$ have the same asymptotic behavior.

Theorem 4.2. *For large sample sizes n and m, the estimators $\hat{R}_{UMVUE}$ and $\hat{R}_{MLE}$ are asymptotically equivalent.*

Proof. The result follows from the limits:

$$
\begin{aligned}
\lim_{n,m\to+\infty} \hat{R}_{UMVUE} =& \begin{cases} \lim_{n,m\to+\infty} \frac{(n-1)(m+1)}{2nm}\frac{\ln Y_{(1)}}{\ln X_{(1)}} & \text{if} \quad X_{(1)} < Y_{(1)}, \\ \lim_{n,m\to+\infty} 1 - \frac{(n+1)(m-1)}{2nm}\frac{\ln X_{(1)}}{\ln Y_{(1)}} & \text{if} \quad X_{(1)} \geq Y_{(1)}, \end{cases} \\
=& \begin{cases} \frac{\ln Y_{(1)}}{2\ln X_{(1)}} & \text{if} \quad X_{(1)} < Y_{(1)}, \\ 1 - \frac{\ln X_{(1)}}{2\ln Y_{(1)}} & \text{if} \quad X_{(1)} \geq Y_{(1)}, \end{cases} = \hat{R}_{MLE}.
\end{aligned}
$$

□

5. SHRINKAGE ESTIMATOR OF R

A shrinkage estimator is constructed based on an initial guess or a preliminary estimate of the parameters of interest, aiming to produce more stable results and reduce estimation errors. This type of estimator was first introduced by Thompson (1968) and has since been studied extensively. For instance, Sharafi et al. (2021) applied the shrinkage estimation method to estimate the parameters of the three-parameter Kappa distribution.

The shrinkage estimator of R takes the general form

$$\hat{R}_{\text{sh}} = \theta\hat{R} + (1-\theta)R_0,$$

where R_0 is an initial guess for R, $\hat{R}$ is an estimator of R, and $\theta \in [0,1]$ is a weight determined through optimization. The optimal value of θ that minimizes the mean squared error (MSE) of $\hat{R}_{\text{sh}}$ is given by

$$\theta = \frac{(R-R_0)(\mathbb{E}[\hat{R}]-R_0)}{\mathbb{E}[\hat{R}^2]+R_0^2-2R_0\mathbb{E}[\hat{R}]}, \tag{7}$$

which depends on the unknown parameter R; see Sharafi et al. (2021) for more details. By substituting $\hat{R}$ for R in the optimal value (7), the shrinkage estimator of R, denoted by $\tilde{R}_{sh}$, is defined as

$$\tilde{R}_{sh} = \theta^*\hat{R} + (1-\theta^*)R_0, \tag{8}$$

where

$$\theta^* = \frac{(\hat{R}-R_0)(E(\hat{R})-R_0)}{E(\hat{R}^2)+R_0^2-2R_0E(\hat{R})}.$$

By considering $\hat{R} = \hat{R}_{MLE}$ and $R_0 = \hat{R}_{UMVUE}$ in the optimum shrinkage estimator (8), we can get a shrinkage estimation for R as follows:

$$\tilde{R}_{sh.1} = \theta_1^*\hat{R}_{MLE} + (1-\theta_1^*)\hat{R}_{UMVUE}, \tag{9}$$

with

$$\theta_1^* = \frac{(\hat{R}_{MLE}-\hat{R}_{UMVUE})(E(\hat{R}_{MLE})-\hat{R}_{UMVUE})}{E(\hat{R}_{MLE}^2)+\hat{R}_{UMVUE}^2-2\hat{R}_{UMVUE}E(\hat{R}_{MLE})}, \tag{10}$$

where from Corollary 3.1,

$$
\begin{aligned}
E(\hat{R}_{MLE}) &= \frac{nm}{2(n+m)(m+1)}\frac{\ln b_2}{\ln b_1} + \frac{m}{n+m} - \frac{nm}{2(n+m)(n+1)}\frac{\ln b_1}{\ln b_2},\\
E(\hat{R}^2_{MLE}) &= \frac{nm}{4(n+m)(m+2)}\left(\frac{\ln b_2}{\ln b_1}\right)^2 + \frac{m}{n+m} - \frac{nm}{(n+m)(n+1)}\frac{\ln b_1}{\ln b_2}\\
&\quad + \frac{nm}{4(n+m)(n+2)}\left(\frac{\ln b_1}{\ln b_2}\right)^2.
\end{aligned}
$$

Since $E(\hat{R}_{MLE})$ and $E(\hat{R}^2_{MLE})$ depend on the parameter b_1 and b_2, we consider the MLEs of these expectations, i.e.

$$
\begin{aligned}
\hat{E}(\hat{R}_{MLE}) &= \frac{nm}{2(n+m)(m+1)}\frac{\ln Y_{(1)}}{\ln X_{(1)}} + \frac{m}{n+m} - \frac{nm}{2(n+m)(n+1)}\frac{\ln X_{(1)}}{\ln Y_{(1)}},\\
\hat{E}(\hat{R}^2_{MLE}) &= \frac{nm}{4(n+m)(m+2)}\left(\frac{\ln Y_{(1)}}{\ln X_{(1)}}\right)^2 + \frac{m}{n+m} - \frac{nm}{(n+m)(n+1)}\frac{\ln X_{(1)}}{\ln Y_{(1)}}\\
&\quad + \frac{nm}{4(n+m)(n+2)}\left(\frac{\ln X_{(1)}}{\ln Y_{(1)}}\right)^2,
\end{aligned}
$$

in (10). Hence, the shrinkage estimator (9) becomes

$$
\tilde{R}^*_{sh.1} = \hat{\theta}^*_1 \hat{R}_{MLE} + (1-\hat{\theta}^*_1)\hat{R}_{UMVUE}, \tag{11}
$$

where

$$
\hat{\theta}^*_1 = \frac{(\hat{R}_{MLE} - \hat{R}_{UMVUE})(\hat{E}(\hat{R}_{MLE}) - \hat{R}_{UMVUE})}{\hat{E}(\hat{R}^2_{MLE}) + \hat{R}^2_{UMVUE} - 2\hat{R}_{UMVUE}\hat{E}(\hat{R}_{MLE})}.
$$

The following theorem addresses the asymptotic behavior of the shrinkage estimator $\tilde{R}^*_{sh.1}$.

Theorem 5.1. *For large sample sizes n and m, the estimators $\hat{R}_{UMVUE}$, $\hat{R}_{MLE}$, and $\tilde{R}^*_{sh.1}$ coincide.*

Proof. The result follows directly from Theorem 4.2 and the definition of $\tilde{R}^*_{sh.1}$ in (11). □

6. SIMULATION STUDY

In this section, we conduct a simulation study to evaluate the performance of the MLE, UMVUE, and the proposed shrinkage estimator of R under different sample sizes. Independent samples $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_m$ are generated from $LU(b_1 = 1.5)$ and $LU(b_2 = 3)$, respectively, with $n,m \in \{25,40,75,116\}$. These parameter values represent distinct stress and strength conditions, and the chosen sample sizes cover a practical range from small to moderately large datasets. Each scenario is replicated 100,000 times to ensure stable Monte Carlo estimates. According to (3), the true value of R is 0.8154649.

Table 1 reports the average bias (A.Bias) and mean squared error (MSE) of the MLE, UMVUE, and the proposed shrinkage estimator of R across different sample sizes.

Table 1: A.Bias and MSE of different estimators of R.

n	m	$\hat{R}_{MLE}$		$\hat{R}_{UMVUE}$		$\tilde{R}^*_{sh.1}$	
		A.Bias	MSE	A.Bias	MSE	A.Bias	MSE
25	25	-0.00027488	0.00011027	0.00002080	0.00010984	0.00002099	0.00010984
	40	0.00025408	0.00007897	-0.00001705	0.00007452	0.00001840	0.00007456
	75	0.00025427	0.00007470	-0.00001531	0.00005587	-0.00001818	0.00005621
	116	0.00024726	0.00007097	-0.00000752	0.00005336	0.00001271	0.00005323
40	25	-0.00296273	0.00009018	0.00003722	0.00007874	0.00005675	0.00007882
	40	-0.00009788	0.00004248	0.00001750	0.00004242	0.00001753	0.00004242
	75	0.00005837	0.00003029	-0.00001138	0.00001665	-0.00001507	0.00002664
	116	0.00002924	0.00003006	-0.00000969	0.00001283	0.00000973	0.00002281
75	25	-0.00021921	0.00009602	0.00002764	0.00006514	-0.00003160	0.00006536
	40	-0.00021222	0.00003373	0.00002745	0.00002814	0.00003013	0.00002815
	75	-0.00003131	0.00001206	0.00001490	0.00001205	0.00001951	0.00001205
	116	0.00003101	0.00000907	-0.00001125	0.00000845	-0.00001261	0.00000845
116	25	-0.00020586	0.00010298	-0.00001164	0.00006185	0.00007068	0.00006214
	40	-0.00011835	0.00003563	-0.00000497	0.00002503	0.00001576	0.00002505
	75	-0.00008901	0.00000958	0.00000427	0.00000871	0.00000653	0.00000871
	116	-0.00000385	0.00000507	0.00000385	0.00000507	0.00000385	0.00000507

The obtained results reveal the following points:

- As the sample sizes increase, the absolute values of A.Bias and the MSE of $\hat{R}_{MLE}$ decrease, confirming the consistency property of the MLE. Similarly, the MSE values of the other estimators also decrease with increasing sample sizes, which numerically supports the consistency of both $\hat{R}_{UMVUE}$ and $\tilde{R}^*_{sh.1}$.

- As expected, $\hat{R}_{UMVUE}$ exhibits the smallest absolute value of average bias among the estimators. Moreover, it is observed that

$$|\text{A.Bias}(\hat{R}_{UMVUE})| < |\text{A.Bias}(\tilde{R}^*_{sh.1})| < |\text{A.Bias}(\hat{R}_{MLE})|.$$

- The MSE values of $\hat{R}_{UMVUE}$ are the lowest compared to those of the other estimators, and in some cases, are equal to those of $\tilde{R}^{*}_{sh.1}$. For each pair (n,m), the following inequality holds:

$$\text{MSE}(\hat{R}_{UMVUE}) \leq \text{MSE}(\tilde{R}^{*}_{sh.1}) < \text{MSE}(\hat{R}_{MLE}).$$

- As expected, the shrinkage estimator $\tilde{R}^{*}_{sh.1}$ outperforms $\hat{R}_{MLE}$ in terms of both absolute value of average bias and MSE. It is worth noting that, since $\hat{R}_{MLE}$ is used as $\hat{R}$ in (8), the shrinkage estimator $\tilde{R}^{*}_{sh.1}$ effectively reduces the bias and MSE compared to $\hat{R}_{MLE}$.

- As the sample sizes increase, the absolute average bias and MSE values of all estimators converge and become very close to each other.

7. APPLICATIONS

This section illustrates the practical applicability of the proposed estimation methods through their implementation in two distinct domains. The first subsection addresses an application in educational psychology, where the estimators are employed to analyze relevant performance measures. The second subsection focuses on electrical engineering, demonstrating the use of these estimators in evaluating stress-strength parameters based on battery lifetime data.

These applications underscore the robustness and adaptability of the MLE, UMVUE, and shrinkage estimators across diverse real-world contexts.

7.1. APPLICATION IN EDUCATIONAL PSYCHOLOGY

Classroom lectures and examinations present significant challenges for students throughout their academic journey. Various factors can influence students energy levels, potentially impairing their ability to maintain focus during lectures and effectively grasp course concepts. Such factors may ultimately contribute to reduced academic performance and lower exam scores.

Numerous studies have examined the impact of caffeine on students' academic performance and examination outcomes. For instance, Aniţei et al. (2011) investigated the influence of energy drink consumption and caffeine intake on reaction time and cognitive processes among young Romanian students. Similarly, Khan et al. (2017) conducted a study to assess the level of caffeine consumption and its effects on academic performance among medical students at Dow University of Health Sciences in Karachi, Pakistan. More recently, Ibn Idris et al.

(2022) explored the effects of caffeine on medical students' exam performance at Northern Border University (NBU), Saudi Arabia.

In this subsection, we examine the effect of coffee consumption, as a source of caffeine, on students performance in statistics exams, based on the results obtained in previous sections. Specifically, we analyze data from two groups of students who took a 10-question statistics exam: one group that did not consume coffee and another that drank two cups prior to the exam. The dataset, available at `https://github.com/mwaskom/Psych252/blob/master/WWW/datasets/caffeine.csv`, includes a potential mediator—students accuracy, measured as the probability of correctly answering a question given their effort. The data are summarized in Table 2.

Table 2: Accuracy scores (potential mediator) of students who consumed either 0 or 2 cups of coffee prior to the statistics exam.

Cups of coffee	Accuracy scores			
2	0.4212120	0.3528293	0.5564924	0.4788779
	0.4047098	0.5580818	0.3838812	0.5772318
	0.6154113	0.6771771	0.2402377	0.6036828
	0.3460965	0.6179977	0.5452024	0.4069703
	0.3953451	0.6337638	0.4618152	0.4105491
0	0.4498767	0.4995339	0.4985903	0.4543119
	0.6262978	0.3501472	0.3988599	0.7308104
	0.6786753	0.6320787	0.6273589	–

Let X denote the potential mediator of accuracy for students who consumed two cups of coffee, and let Y represent the corresponding mediator for students who did not consume coffee. Our goal is to investigate the impact of coffee consumption on students accuracy by estimating the probability that drinking coffee increases this mediator, i.e., $R = P(X > Y)$.

As a preliminary step, we assess whether the log-uniform distribution provides an appropriate fit for the observed data. To this end, we estimate the parameters b_1 and b_2 using the maximum likelihood method and perform the Kolmogorov-Smirnov (KS) goodness-of-fit test. Table 3 presents a summary of the results.

Based on the results presented in Table 3, we conclude that the log-uniform distribution provides an adequate fit for the observed data. Consequently, using (4) and Theorem 4.1, the MLE and UMVUE of $R = P(X > Y)$ are $\hat{R}_{MLE} = 0.3679204$ and $\hat{R}_{UMVUE} = 0.3177494$, respectively. Furthermore, applying (11), the shrink-

Table 3: KS test results and MLEs of the parameters for student accuracy data with and without coffee consumption.

Data of students with drinking	MLE	test statistic	p-value
2 cups of coffee	$\hat{b}_1 = 4.162543$	0.27334	0.08224
0 cups of coffee	$\hat{b}_2 = 2.855942$	0.29884	0.22900

age estimation of R is given by

$$\tilde{R}^*_{sh.1} = \hat{\theta}^*_1(0.3679204) + \left(1 - \hat{\theta}^*_1\right)(0.3177494) = 0.3588493,$$

where $\hat{\theta}^*_1 = 0.8191972$.

The calculated values of the different estimators of R indicate that the probability of coffee consumption leading to improved accuracy scores is approximately 0.3. This finding suggests that drinking coffee prior to a statistics exam may enhance student performance. Caffeine is known to increase energy levels, improve concentration, and enhance cognitive functioning, which may collectively contribute to better exam outcomes. However, it is important to note that the extent of caffeines effect on academic performance may vary depending on individual physiological responses.

These findings align with a substantial body of literature reporting beneficial effects of caffeine on cognitive performance and memory. For example, Sherman et al. (2016) demonstrated that caffeine intake can significantly enhance explicit memory performance in young adults during morning testing sessions, suggesting a time-of-day-dependent cognitive benefit.

7.2. APPLICATION IN ELECTRICAL ENGINEERING

Reliability analysis plays a crucial role in electrical engineering, particularly in evaluating the long-term performance and failure behavior of components and systems. A key aspect of reliability modeling is estimating the probability that a system's inherent strength exceeds the operational stress, which is essential for predicting system reliability and ensuring safe and efficient designs.

In this subsection, we apply the proposed estimators to analyze the degradation behavior of lithium-ion batteries subjected to different thermal storage conditions. Specifically, we use real-world capacity data (in Ah) collected over a 3-week period from batteries stored at two different temperatures: 25°C (representing low-stress or room-temperature conditions) and 50°C (representing high-stress or elevated-temperature conditions). The dataset, obtained from the CALCE battery data repository (https://calce.umd.edu/battery-data), provides an opportunity to assess how temperature-induced stress impacts battery capacity and overall reliability.

In this context, we define *strength* as the remaining capacity of a battery stored under room-temperature conditions (25°C), and *stress* as the remaining capacity of a battery exposed to harsher conditions (50°C). Under this framework, the reliability is given by $R = P(X > Y)$ where X and Y represent the capacities of batteries from the 25°C and 50°C groups, respectively. The collected capacity data for both conditions is summarized in Table 4.

Table 4: Charge capacity (Ah) data of lithium-ion batteries stored at two different temperatures: 25°C and 50°C.

Temperature	Charge Capacity			
25°C	0.060511487	0.200307983	0.139800208	0.006259688
	0.022952455	0.131453684	0.166926078	0.137713617
	0.171099205	0.004173094	0.010432821	0.064684697
	0.039645317	0.116847193	0.062598126	0.075117785
50°C	0.002086253	0.168988979	0.181507477	0.039639947
	0.098051629	0.054244232	0.006258799	0.177334650
	0.014603951	0.018776534	0.208630818	0.010431352
	0.131433696	0.143952122	0.112656079	0.123088079

To assess the suitability of the log-uniform distribution for modeling the battery capacity data, we conducted the KS tests. As reported in Table 5, the test results indicate that the log-uniform distribution adequately describes the data. Based on the estimated values of b_1 and b_2 provided in Table 5, the MLE of R is 0.5561605. Similarly, the UMVUE and the shrinkage estimation of R are 0.5578942 and 0.5578908 (with $\hat{\theta}_1^* = 0.0019926$), respectively.

These results suggest that, under the stress-strength model framework, the probability that a battery stored at room temperature (25°C) maintains a higher remaining capacity than one stored at elevated temperature (50°C) is approximately 0.56. This indicates that elevated thermal storage conditions significantly accelerate degradation compared to mild conditions.

These findings are consistent with extensive literature documenting the effects of elevated temperature on lithium-ion battery degradation. For instance, Li et al. (2023) recently reported that thermal gradients as small as 3řC between active cell regions can accelerate degradation by over 300% compared to uniform temperature conditions. Also, Shen et al. (2022) highlight that high operating

Table 5: KS test results and MLEs of the parameters for the battery capacity data at two storage temperatures.

Temperature	MLE	test statistic	p-value
25°C	$\hat{b}_1 = 239.6304$	0.29346	0.1025
50°C	$\hat{b}_2 = 479.3283$	0.25390	0.2137

temperatures significantly compromise both capacity retention and safety, particularly through mechanisms such as increases in solid-electrolyte interface (SEI) growth and heat generation.

Our estimated probability $R \approx 0.56$ aligns with these thermal degradation dynamics, reflecting a moderate but measurable risk: batteries stored under elevated temperature (50°C) are distinctly more likely to exhibit lower remaining capacity than those stored at mild temperature (25°C). The close numerical agreement between the MLE, UMVUE, and the shrinkage estimator further confirms the robustness and consistency of the proposed estimation methods for real-world reliability applications.

CONCLUSION

In this paper, we derived and analyzed the stress-strength parameter for log-uniform distributions, obtaining its maximum likelihood estimator, uniformly minimum variance unbiased estimator, and a shrinkage estimator. A comprehensive simulation study was conducted to evaluate the performance of these estimators. The results demonstrated that, while the UMVUE generally exhibited the smallest bias and mean squared error, all three estimators converged and performed similarly as sample sizes increased, confirming their asymptotic equivalence.

To illustrate the practical applicability of our results, we applied the proposed estimators to two real-world problems:

- an educational psychology setting, analyzing the effect of coffee consumption on students performance in statistics exams, and
- an electrical engineering context, evaluating the degradation behavior of lithium-ion batteries under different thermal storage conditions.

In the educational application, the estimated probability that coffee consumption improves exam performance was approximately 0.35, indicating a modest positive effect. In the engineering application, the estimated parameter suggested a moderate but measurable risk of capacity loss for batteries stored at elevated temperatures.

Although this study focused on frequentist estimation methods, the proposed framework can be naturally extended to Bayesian inference or other estimation approaches. Moreover, the methodology has potential applications in various fields where stress-strength models arise, beyond the two domains considered here.

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CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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