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Normalizing Risk Measures in Risk-Based Portfolios through Covariance Misspecification Error Analysis

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Abstract *This paper focuses on evaluating allocation strategies in portfolio management, specifically examining methods for determining asset weights. The study emphasizes the covariance matrix, a critical component in constructing risk-based portfolios, including minimum volatility, inverse volatility, equal risk contribution, and maximum diversification portfolios. The primary aim is to analyze the robustness and sensitivity of these strategies under potential misspecifications or errors in the covariance matrix. Using a Dynamic Conditional Correlation model and a Monte Carlo simulation approach, a large set of covariance matrices is generated. Risk-based allocation strategies are then applied to these simulated matrices, and robustness is assessed by quantifying deviations between actual and simulated allocations. Furthermore, the study estimates the probability of model accuracy and incorporates this into two conventional risk measures. These adjusted measures account for the risk of covariance misspecification, providing a normalized and more reliable evaluation of portfolio performance. This approach enhances the interpretability and robustness of risk metrics in the presence of estimation errors, offering valuable insights for portfolio optimization under realistic uncertainty conditions.*

Keywords: *Dynamic Conditional Correlation Model, Risk Measure Adjustment, Asset Allocation, Covariance Misspecification, Risk-based Portfolios.*

1. INTRODUCTION

Many modern portfolio optimization methodologies are grounded in the seminal mean-variance framework proposed by Markowitz (1952), which forms the cornerstone of modern portfolio theory. However, practical implementation of mean-variance optimization is often challenged by inaccuracies in estimating the sample mean, particularly in scenarios involving small sample sizes or high-dimensional datasets (Michaud, 1989). To address these limitations, risk-based portfolio strategies have emerged, focusing exclusively on the estimation of the covariance matrix, thereby mitigating estimation errors associated with mean returns (de Miguel et al., 2007). Examples of such approaches include minimum-variance portfolios (Clarke et al., 2006), risk parity portfolios (Maillard et al., 2010), and maximum diversification portfolios (Choueifaty and Coignard, 2008). These methodologies emphasize robustness and are particularly attractive to investors for their capacity to prioritize diversification and effective risk management, providing a practical alternative to strategies heavily reliant on mean return estimation (Thierry, 2013).

However, a new challenge arises in minimizing allocation errors across these risk-based portfolios, particularly when facing misspecification of the covariance matrix, a

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well-documented concern in portfolio optimization literature (Bickel and Levina, 2008; Ledoit and Wolf, 2003). This issue becomes especially pronounced with an increasing number of securities in the portfolio, as the estimation error of the covariance matrix grows with dimensionality (Fan et al., 2016). Misspecification in asset allocation typically occurs when flawed or inaccurate model inputs are used to compute asset weights within an investment portfolio, leading to suboptimal performance (Chopra and Ziemba, 1993).

In this study, we focus on four risk-based strategies: minimum volatility (Clarke et al., 2006), inverse volatility (Thierry, 2013), equal risk contribution (ERC) (Maillard et al., 2010), and maximum diversification (Choueifaty and Coignard, 2008) portfolios. By systematically analyzing and quantifying the potential errors resulting from inaccurate estimation of the covariance matrix, we aim to gain insights into their implications for asset allocation decisions. This assessment enables us to evaluate the reliability and robustness of various risk-based strategies, providing a foundation for informed adjustments to mitigate the impact of potential misspecifications. Our work contributes to the growing body of research addressing the practical limitations of risk-based optimization approaches under real-world estimation constraints (Ledoit and Wolf, 2004).

We consider a covariance matrix constructed from n assets, represented as a square $n \times n$ matrix, where each row and column correspond to a specific asset. The diagonal elements of the matrix denote the variances of individual asset returns, reflecting the level of volatility or risk associated with each asset. The off-diagonal elements capture the covariances between pairs of assets, indicating the degree to which their returns co-move. Misspecification in the covariance matrix implies inaccuracies in the estimated relationships between asset returns and volatilities. Such errors distort portfolio optimization by leading to flawed asset allocations, potentially resulting in suboptimal portfolio configurations. Specifically, inaccuracies in variance and covariance estimates may cause over- or under-allocation of capital to certain assets, thereby either exposing the portfolio to unintended risks or limiting potential returns. Addressing these estimation errors is critical for achieving robust and efficient portfolio solutions.

Our study aims to partially replicate and extend the findings of Ardia et al. (2017a), which investigates the impact of covariance matrix inaccuracies on the determination of optimal weights in risk-based portfolio optimization strategies. In the first phase, we assess the robustness of various risk-based strategies under inaccurate input conditions by employing the L-distance metric to quantify allocation errors. Subsequently, we integrate the L-distance into the formulation of novel risk-adjusted measures. These metrics account for the robustness of models when flawed inputs are present, thereby providing more reliable risk estimates that incorporate the uncertainty associated with input inaccuracies. To validate this approach, we conduct experiments across multiple investment universes, constructing portfolios with assets categorized by economic sectors, asset classes, prominent large-cap technology firms, and leading financial industry players. Our findings reveal that different risk-based strategies exhibit varying sensitivities to imprecise inputs. Specifically, the Minimum-Volatility (Min-Vol) and Maximum Diversification (Max-Div) strategies demonstrate heightened vulnerability, as their allocation processes tend to concentrate on a small number of assets. This results in significant discrepancies between true and misspecified allocations, underscoring the need for robust measures in practical applications.

The rest of the paper is structured as follows: Section 2 provides the necessary background to the issue of covariance misspecification, Section 3 introduces the four risk-based strategies considered in the empirical analysis. Section 4 presents the model misspecification risk metrics whilst Section 5 describes the data utilized. Section 6 summarizes the empirical results, and Section 7 provides the conclusions.

2. BACKGROUND

Our primary goal is evaluating the robustness of risk-based investment strategies by considering a novel risk measure called the L-distance, as introduced by Ardia et al. (2017a). This measure quantifies the discrepancy between the true portfolio weights and those derived from simulations using misspecified volatility matrices as inputs.

To simulate covariance matrices in a Monte Carlo framework, we employ the Dynamic Conditional Correlation (DCC) model, a methodology introduced by Engle (2002). The DCC model estimates time-varying variances and correlations using parameters analogous to those in a GARCH(1,1) process, enabling the dynamic capture of persistence in both variance and correlation structures. This facilitates the simulation of realistic covariance matrices while defining a rebalancing horizon. For practical implementation, we utilize the R package rmgarch (Galanos, 2022), which provides an efficient and robust implementation of the DCC-GARCH model. Notably, this package was previously employed in the misspecification study by Ardia et al. (2017a).

Our work closely aligns with the study by Yaoxiang (2018), which investigates the sensitivity of Value at Risk (VaR) (Duffie and Pan, 1997) to misspecifications in variance and covariance terms within risk-based portfolio construction. Similar to our approach, they utilize a multivariate GARCH model to simulate misspecified inputs through the Monte Carlo method. However, their primary focus lies in assessing the sensitivity of portfolio VaR to inaccuracies in the covariance matrices, rather than modifying the risk measure itself to account for such errors. In addressing covariance misspecification, Peterson and Grier (2006) proposed an enhanced covariance estimator tailored for short return series. Their approach, akin to ours, seeks to mitigate the adverse effects of input inaccuracies to prevent the construction of sub-optimal portfolio allocations. Another relevant study is that of Neffelli (2018), who employed extensive Monte Carlo simulations to evaluate target estimators. Their comparative analysis assessed the accuracy of these estimators in reproducing true portfolio weights while accounting for dataset dimensionality and shrinkage intensity across four risk-based portfolios. Neffelli's introduction of the shrinkage method effectively reduced misspecification errors in the target covariance matrix, providing more reliable inputs for portfolio allocation. By contrast, Jain and Jain (2019) analyzed the impact of covariance matrix misspecification on portfolio performance, comparing traditional risk-based allocation methods with machine learning techniques. Although the methodologies differ, both approaches fundamentally depend on accurate covariance matrix estimation; any misspecification can result in suboptimal portfolio allocations.

While the aforementioned studies focus primarily on covariance estimation errors, our study advances the literature by proposing novel risk measures that explicitly incorporate the costs associated with model misspecification. Furthermore, we introduce a normalization process for conventional risk measures within risk-based strategies, providing a more robust and theoretically grounded framework for portfolio construction.

3. RISK-BASED STRATEGIES

Risk-based strategies are investment methods used to allocate assets within a financial portfolio. Unlike traditional approaches that estimate mean returns, these strategies rely solely on the covariance matrix as input. The strategies discussed will be tested in the context of covariance misspecification. They are implemented using the following equations:

1. **The minimum variance (MV) portfolio:** The goal of this strategy is to build a portfolio with minimal volatility (Markowitz, 1952). The allocations within this portfolio are determined are:

$$\mathbf{w}_{\min} = \arg\min_{w \in C} \left(w' \Sigma w\right) \tag{1}$$

   where C represents the long-only full investment constraint that sums to one, $\mathbf{w}$ is the vector of the weights, and $\mathbf{\Sigma}$ is the covariance-variance matrix.

2. **The inverse-volatility (IV) portfolio:** This strategy constructs a portfolio by allocating investments based on the inverse of each asset's historical volatility. It is designed to give higher weightings to assets with lower historical volatility and lower weightings to those with higher historical volatility (de Carvalho et al., 2012). The IV portfolio weights are defined as follows:

$$\mathbf{w_{iv}} = \left(\frac{1/\sigma_1}{\sum_{i=1}^{N} 1/\sigma_j} \cdots \frac{1/\sigma_N}{\sum_{i=1}^{N} 1/\sigma_j} \right) \tag{2}$$

   where σ is the volatility of each stock considered, $\mathbf{w_{iv}}$ is the vector of the weights, and N is the number of stocks considered.

3. **The equal-risk contribution (ERC) portfolio:** This strategy seeks to construct a portfolio with the highest possible level of diversification (Choueifaty and Coignard, 2008). The portfolio's diversification ratio is defined as:

$$\%RC = \frac{w_i [\mathbf{\Sigma w}]_\mathbf{i}}{\mathbf{w' \Sigma w}} \tag{3}$$

   The ERC weights are then computed as:

$$\mathbf{w_{erc}} = \arg\min_{w \in C} \left\{ \sum_{i=1}^{N} (\%RC_i - \frac{1}{N})^2 \right\} \tag{4}$$

4. **The maximum diversification (MD) portfolio:** This risk-based strategy aims to construct a portfolio with the highest possible level of diversification (Choueifaty and Coignard (2008)). Considering the portfolio's diversification ratio as:

$$DR(\mathbf{w}) = \frac{\mathbf{w}'\sigma}{\sqrt{w'\mathbf{\Sigma}\mathbf{w}}} \geq 1 \tag{5}$$

The MD weights are computed using the formula:

$$\mathbf{w_{md}} = \arg\max_{w \in C} \ DR(\mathbf{w}) \tag{6}$$

In practice, the ERC strategy aims to allocate portfolio weights such that each asset contributes equally to the total portfolio risk. The core idea behind this strategy is that no single asset dominates the portfolio's overall risk. This approach emphasizes risk parity, distributing the risk contribution from each asset equally, rather than focusing on returns or absolute weight allocation. Mathematically, it seeks to balance the marginal risk contributions of each asset, accounting for the covariance structure of asset returns. Instead, the MD strategy aims to maximize the diversification ratio, defined as the ratio of the weighted sum of asset volatilities to the portfolio volatility. The goal is to achieve the maximum spread of risk across assets, effectively increasing the benefit from diversification. This strategy optimizes portfolio weights to enhance the risk-return trade-off while leveraging the imperfect correlation between asset returns.

All of these investment strategies utilize elements of the covariance matrix as inputs to define portfolio allocations. These four strategies will be subjected to testing under a scenario of misspecified inputs, with the objective of evaluating their robustness in handling errors.

We will conduct the portfolio optimization process using the R package 'RiskPortfolios' developed by Ardia et al. (2017b), which calculates the portfolio weights for each of the four risk-based allocations discussed in this section.

4. PROPOSED APPROACH

To analyze real-world portfolios and address potential misspecification issues in asset allocation, our work is organized into the following phases:

- First, we define the specific time range and select the portfolios for the analysis. The return data is modeled using a Dynamic Conditional Correlation (DCC) model to estimate the relevant parameters. The DCC model captures the time-varying correlations among asset returns, providing essential inputs for scenario generation. These parameters are then utilized to perform Monte Carlo simulations, ensuring that the simulated scenarios accurately reflect the statistical properties and dynamics of the observed data.

- Next, we simulate covariance matrices under the assumption of daily rebalancing, corresponding to a periodicity of $T+h$ with $h=1$. This approach enables us to forecast one period ahead and generate a large ensemble of simulated covariance matrices. The arithmetic mean of these simulated matrices is treated as the 'true' covariance matrix. This procedure allows us to compare the optimal allocations derived from the average covariance matrix (representing the true covariance) with the allocations obtained from each individual Monte Carlo simulation.

- The simulated covariance matrices will inevitably deviate from the true covariance matrix, offering a framework to quantify the misspecification inherent in each risk-based portfolio strategy. By evaluating the discrepancies in portfolio weights between the allocations derived from the true covariance matrix and those obtained using the simulated matrices, we can quantitatively measure the extent of misspecification. This analysis provides critical insights into the influence of covariance matrix misspecification on the performance and robustness of the strategies under study. Notationally, the degree of misspecification L is defined as

$$L = |\mathbf{w} - \hat{\mathbf{w}}_{\mathbf{i}}| \tag{7}$$

  where $\mathbf{w}$ represents the vector of portfolio weights derived from the true covariance matrix, and $\hat{\mathbf{w}}_{\mathbf{i}}$ denotes the weights obtained from the i-th simulated covariance matrix. The L distance quantifies the deviation between the true and simulated weights, serving as a measure of misspecification error. A larger L implies greater sensitivity of the strategy to inaccuracies in the covariance matrix, thereby indicating lower robustness. To systematically assess this misspecification, we generate over 500 distinct Monte Carlo covariance matrices for each investment universe. These matrices facilitate the computation of misspecification distances and enable a robust evaluation of the stability and reliability of each risk-based portfolio strategy in the presence of covariance matrix uncertainty.

- In the final phase, we utilize the misspecification risk measure to enhance existing risk metrics or construct novel ones, aiming for a more accurate characterization of the inherent risks associated with each risk-based strategy. Initially, we apply the risk-based portfolio approach to determine a set of allocations derived from historical returns. Subsequently, we perform daily rebalancing of these allocations to compute conventional risk measures, including annualized return, annualized standard deviation, Sharpe ratio, and Pain Index, which collectively evaluate the performance of each strategy. Building on these conventional measures, we introduce modifications and develop new metrics that incorporate the L distance into their formulation.

4.1. MODEL MISSPECIFICATION RISK MEASURES

4.1.1. COST OF INCORRECT MODEL SPECIFICATION

To better quantify the cost associated with incorrect model specification, we introduce two indices derived using the L distance as a reference metric:

- **Ret/L index**: This index represents the annualized return per unit of L distance error. A higher value indicates greater returns achieved relative to the degree of model misspecification. It is defined as:

$$Ret/L = \frac{Annualized\ Return}{L\ distance} \tag{8}$$

  A higher Ret/L index implies a more favorable trade-off, where higher returns are obtained with relatively lower risk due to misspecification. Investors typically prefer strategies with higher Ret/L values as they maximize returns while minimizing the impact of model error.

- **Std/L index**: This index captures the annualized volatility per unit of L distance error. A higher value reflects increased portfolio volatility associated with an additional unit of model misspecification. It is computed as:

$$Std/L = \frac{Annualized\ Standard\ Deviation}{L\ distance} \tag{9}$$

  Portfolios with higher Std/L values experience greater volatility for the same level of model misspecification. Risk-tolerant investors might favor such portfolios if they view higher volatility as a potential for greater returns despite the inherent model error.

It is crucial to emphasize that the interpretation of these indices is context-dependent and influenced by the investor's specific risk preferences. While some investors may prioritize minimizing volatility, others may focus more on addressing the potential consequences of model misspecification. These ratios serve as a unified metric, integrating both dimensions of risk, thereby enabling a more holistic and nuanced evaluation of investment performance.

4.1.2. CONVENTIONAL RISK MEASURES NORMALIZED

In this section, we propose an adjustment framework for conventional risk measures to integrate the impact of misspecification risk associated with the implemented strategy. Specifically, the objective is to normalize the risk measure to systematically account for the inherent vulnerabilities and strengths of the underlying risk-based strategy that informs its estimation.

To achieve this adjustment, we begin by calculating the probability of accurately estimating the covariance matrix for a given strategy. This probability is then multiplied by the conventional risk measure, yielding an adjusted risk measure. The adjusted measure incorporates an enhanced representation of the risk indicator, reflecting not merely the standard metric but one that accounts for the likelihood of accurate input estimation, thereby improving its practical relevance and interpretative value.

The probability of estimating a covariance matrix with specification errors, denoted as P_e, is:

$$\begin{aligned} P_e = & \frac{Value\ of\ the\ specification\ error}{Maximum\ total\ specification\ error\ possible} = \\ = & \frac{L\ distance \times 100}{100 \times (n_{assets} - 1)} = \\ = & \frac{|\mathbf{w} - \hat{\mathbf{w}}_\mathbf{i}| \times 100}{100 \times (n_{assets} - 1)} \end{aligned} \tag{10}$$

In Equation 10, the numerator quantifies the L distance associated with the strategy, computed as the absolute difference between the historical and simulated portfolio weights. This value is scaled by a factor of 100 to express the L distance as a percentage, enhancing interpretability in relative terms. The denominator is fixed at 100, corresponding to the percentage of the maximum allocation permissible in a single asset under a long-only allocation strategy. This value is further scaled by the number of assets minus one, representing the maximum potential reallocation if the simulated weights are entirely misspecified. Accordingly, the probability of correctly estimating the model is:

$$P_a = 1 - P_e \tag{11}$$

As the misspecification error increases (i.e., as the L distance grows), the numerator in the expression for the probability of estimating an incorrect model (P_e) also increases. Consequently, this leads to a reduction in the probability of correctly estimating the model (P_a). Intuitively, a larger L distance corresponds to a lower likelihood of accurately estimating the allocations under a risk-based strategy.

By integrating the probability P_a with a standard risk measure, a normalized risk metric can be formulated. This metric introduces a probabilistic adjustment to conventional risk evaluation, thereby enhancing the robustness and interpretability of the risk assessment framework:

$$Risk\ Measure\ Normalized = Conventional\ Risk\ Measure \times P_a \tag{12}$$

To operationalize the normalization process, two conventional risk measures that represent portfolio performance using a risk-based strategy are augmented. The novel normalized measures are:

- The **'Normalized Sharpe Ratio'** serves as a risk measure adjusted for the likelihood of accurate model specification. Incorporating the probability of correct model estimation, denoted as (P_a), the adjusted risk measure is calculated as:

  $$Normalized\ Sharpe\ Ratio = \frac{R_p - R_f}{\sigma_p} \times P_a \tag{13}$$

  with R_f being the return of a risk-free investment and R_p and σ_p being the return and the standard deviation of the portfolio, respectively. Eq. 13 defines an augmented Sharpe Ratio, which modifies the conventional Sharpe Ratio by incorporating a weighting factor (P_a) based on the probability of accurately estimating the model. This adjustment reflects the reliability of the underlying strategy. Portfolios with higher values of this risk measure are preferred by investors, as they indicate a greater probability of achieving the expected returns with reduced uncertainty, thereby enhancing the robustness of the investment decision.

- The **'Normalized Pain Index'** indicates the average maximum drawdown normalized by the probability of obtaining this value with certainty. Similar to the Normalized Sharpe Ratio, we consider this risk measure in a context where the model is correctly estimated. Thus, we multiply the Pain Index Bacon (2012) by the probability of accurately estimating its value. The probability of accuracy (P_a) is given by:

$$Normalized\ Pain\ Index = \sum_{i=1}^{i=n} \frac{|D_i'|}{n} \times P_a \tag{14}$$

  where n is the number of observations of the entire series and D_i' is the drawdown since previous peak in period i. This risk measure quantifies the average magnitude of negative returns from each peak in the portfolio's performance. A lower value of this measure is desirable for investors, as it reflects smaller average drawdowns and greater portfolio resilience. The measure is normalized by incorporating the probability of achieving this value with certainty, assuming an accurately specified covariance matrix. Accurate estimation is critical, as any misspecification in the covariance matrix may result in suboptimal portfolio allocations and, consequently, a deviation in the computed Pain Index. This highlights the sensitivity of the measure to estimation accuracy and the importance of robust modeling.

5. DATA

To estimate our model using empirical data, we utilize daily return data spanning a five-year period from January 2015 to January 2020, comprising a total of 1,259 observations. These return data are obtained from the Yahoo Finance database. The analysis focuses on four distinct investment universes, each characterized by the following criteria:

- **The 'Five Industry Portfolio':** This portfolio represents distinct sectors of the American stock market, comprising five Exchange-Traded Funds (ETFs) that correspond to different economic sectors. The ETFs included in this portfolio are *The Financial Select Sector SPDR Fund* (XLF), *The Industrial Select Sector SPDR Fund* (XLI), *The Technology Select Sector SPDR Fund* (XLK), *The Health Care Select Sector SPDR Fund* (XLV), and *The Consumer Discretionary Select Sector SPDR Fund* (XLY).

- **The 'Core Portfolio':** This investment universe consists of six diverse assets across various asset classes, offering significant diversification potential. Each asset is represented by an Exchange-Traded Fund (ETF), providing exposure to different segments of the financial markets. The selected assets for this portfolio

Table 1: Investment universe characteristics

Investment Universe	N	Annualized Volatility			Correlation		
		Min	Med	Max	Min	Med	Max
Five Industry Portfolios	5	0.15	0.15	0.18	0.66	0.78	0.85
Core Portfolio	6	0.12	0.14	0.19	-0.38	0.37	0.88
GAFAM Portfolio	5	0.23	0.25	0.29	0.46	0.60	0.66
Financial Sector Portfolio	6	0.2	0.21	0.26	0.40	0.57	0.89

are *Invesco DB Commodity Index Tracking Fund* (DBC), *iShares MSCI Emerging Markets ETF* (EEM), *SPDR Gold Shares* (GLD), *SPDR S&P 500 ETF Trust* (SPY), *iShares 20+ Year Treasury Bond ETF* (TLT), and *Vanguard Small-Cap Value Index Fund ETF Shares* (VBR).

- **The 'GAFAM' Portfolio:** This portfolio includes a group of five prominent and influential technology companies, namely it Apple (AAPL), *Amazon.com* (AMZN), *Alphabet* (GOOG), *Meta Platforms* (META), and *Microsoft Corporation* (MSFT).

- **The 'Financial Sector Portfolio':** This portfolio consists of a selection of leading companies within the banking and financial services sector. The assets included are *Bank of America Corporation* (BAC), *HSBC Holdings* (HSBC), *JPMorgan Chase & Co.* (JPM), *Mastercard Incorporated* (MA), *Visa* (V), and *Wells Fargo & Company* (WFC).

Table 1 presents summary statistics for the investment universes described above, evaluated across multiple key metrics.

Each row corresponds to a distinct investment universe, while the columns represent metrics such as annualized volatilities and pairwise correlations. For each metric, the table reports the minimum, median, and maximum values observed over the 2010-2015 period.

The inclusion of diverse asset types across these portfolios results in varying levels of portfolio variance and correlation structures, thereby shaping distinct investment landscapes and offering heterogeneous risk-return profiles for potential investors.

6. RESULTS

6.1. THE MISSPECIFICATION ERROR

Table 2 presents the magnitude of the misspecification problem observed when applying different investment strategies across the four considered investment universes. The average L distances were derived from over 500 Monte Carlo simulations of the covariance matrices. The results are presented collectively to enable a direct comparison, highlighting which strategy exhibits greater sensitivity to covariance matrix misspecifications.

An investor faces varying degrees of risk stemming from allocation inaccuracies, which depend on the selected risk-based strategy. Table 2 illustrates that the Minimum

Table 2: **Level of misspecification for each investment universe considered**

Investment Universe	Average L distance			
	Min-Vol	Inv-Vol	ERC	Max Div
Five Industry Portfolios	19.23	1.66	1.61	5.19
Core Portfolio	10.65	1.21	1.23	4.30
GAFAM Portfolio	23.61	2.63	2.60	3.26
Financial Sector Porfolio	13.65	1.76	1.77	5.02

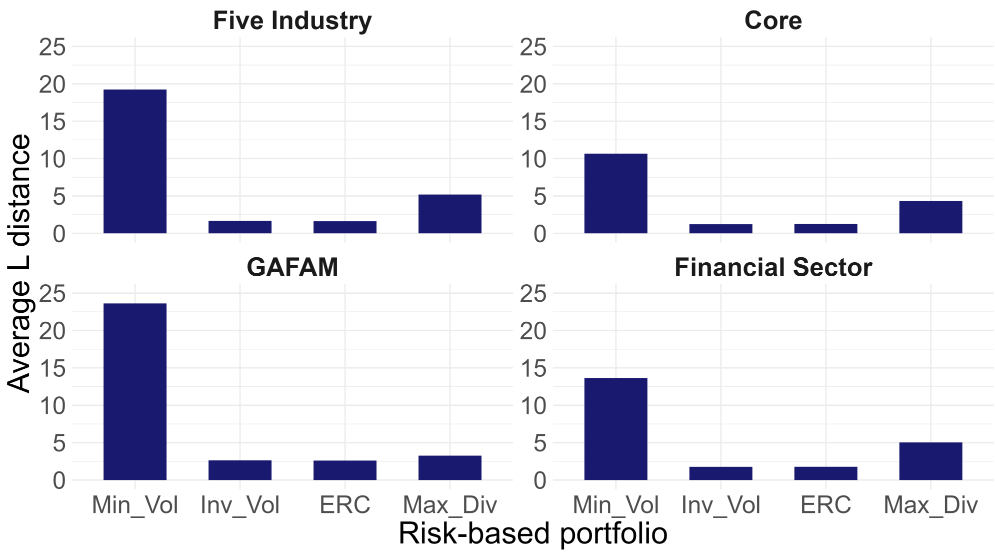

Figure 1: **Level of misspecification in each investment universe (in %)**

Volatility and Maximum Diversification strategies exhibit heightened sensitivity to errors in estimating covariance and variance compared to other risk-based approaches. These findings are visually represented in Figure 1, offering a clearer depiction of the misspecification impact across strategies.

In Figure 1, the Minimum Volatility strategy emerges with the highest sensitivity levels, indicating substantial allocation errors when the covariance matrix is misspecified. This heightened sensitivity arises because the strategy tends to concentrate portfolio weights in a limited subset of assets, amplifying the impact of small estimation errors and potentially redistributing allocations to unintended assets.

Conversely, the Inverse Volatility and Equal Risk Contribution strategies display greater robustness and resilience against inaccuracies in covariance matrix estimation. These approaches yield relatively stable and consistent portfolio allocations, even in the presence of minor input distortions, underscoring their suitability for environments characterized by estimation uncertainty.

It is noteworthy that, across diverse investment universes, the results exhibit con-

Table 3: Risk Indexes using L distances for all investment universes

Universes	Indexes	Min Vol	Inv Vol	Equal Risk	Max Div
Five Industries	***Ret/L***	0.0052	0.0703	0.0724	0.0219
	Std/L	0.0071	0.0850	0.0878	0.0274
Core	***Ret/L***	0.0041	0.0366	0.0328	0.0091
	Std/L	0.0061	0.0644	0.0562	0.0151
GAFAM	***Ret/L***	0.0112	0.1137	0.1147	0.0903
	Std/L	0.0086	0.0788	0.0797	0.0638
Financial	***Ret/L***	0.0067	0.0826	0.0815	0.0218
	Std/L	0.0119	0.0975	0.0965	0.0329

sistent patterns, suggesting that the investment environment exerts minimal influence on the occurrence of allocation errors. In the subsequent sections, we will address the misspecification risk, quantified by the L distance, through the development of novel indicators and adjustments to risk measure values.

6.2. THE COST OF COVARIANCE SPECIFICATION ERRORS

Table 3 presents the values of the performance ratios defined according to Equations 8 and 9.

When assessing annualized returns and standard deviation per unit of misspecification across different strategies, the following observations are noteworthy:

- **Minimum Volatility**: This strategy consistently generates lower returns but also exhibits reduced volatility per unit of misspecification across all investment universes. The misspecification measure L in the denominator remains consistently higher for this strategy, rendering it less appealing for investors particularly sensitive to model misspecification risk.

- **Inverse Volatility** and **Equal Risk Contribution**: These strategies reliably deliver higher returns and greater volatility per unit of misspecification compared to the Minimum Volatility strategy. Both approaches demonstrate robust allocation performance across diverse investment universes. Notably, their resilience to moderate deviations in model inputs ensures stability in allocation outcomes, instilling confidence among investors in the reliability of these strategies.

- **Maximum Diversification**: This strategy occupies an intermediate position in terms of return and volatility per unit of misspecification. While it exhibits greater robustness against covariance matrix errors compared to the Minimum Volatility strategy, it does not match the resilience demonstrated by the Inverse Volatility and Equal Risk Contribution strategies.

In summary, although each investment universe possesses unique characteristics, the Inverse Volatility and Equal Risk Contribution strategies generally outperform in terms of return and volatility per unit of misspecification. The Minimum Volatility strategy

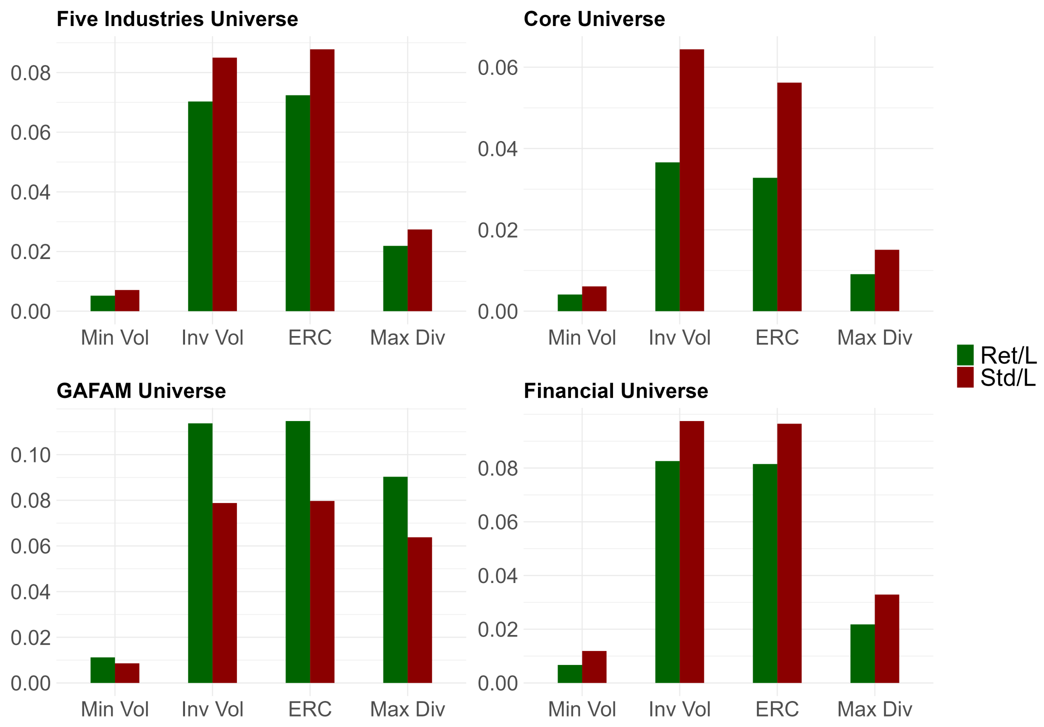

Figure 2: Cost of model misspecification in terms of annualized return and volatility

Table 4: Probability of incorrect estimation of the covariance matrix (in %)

	Probability of error (P_e)			
Universe	**Min Vol**	**Inv Vol**	**Equal Risk**	**Max div**
Five Industries	4.81%	0.42%	0.4%	1.3%
Core	2.13%	0.24%	0.25%	0.86%
GAFAM	5.9%	0.66%	0.65%	0.82%
Financial	2.73%	0.35%	0.35%	1.01%

Table 5: Conventional and normalized risk measures for all investment universes

Universe	Index	Min Vol	Inv Vol	Equal Risk	Max Div
Five Industries	*Sharpe Ratio*	0.0487	0.0538	0.0536	0.0523
	Sharpe Ratio N.	0.0464	0.0535	0.0534	0.0516
	Pain Index	0.0293	0.0286	0.0288	0.0316
	Pain Index N.	0.0279	0.0285	0.0287	0.0312
Core	*Sharpe Ratio*	0.0441	0.0375	0.0382	0.0395
	Sharpe Ratio N.	0.0432	0.0374	0.0381	0.0392
	Pain Index	0.0332	0.0398	0.0383	0.0334
	Pain Index N.	0.0325	0.0397	0.0382	0.0331
GAFAM	*Sharpe Ratio*	0.0794	0.0860	0.0859	0.0847
	Sharpe Ratio N.	0.0747	0.0855	0.0853	0.0840
	Pain Index	0.0364	0.0338	0.0340	0.0358
	Pain Index N.	0.0343	0.0336	0.0338	0.0355
Financial	*Sharpe Ratio*	0.0391	0.0553	0.0551	0.0449
	Sharpe Ratio N.	0.0380	0.0551	0.0549	0.0444
	Pain Index	0.0542	0.0437	0.0436	0.0500
	Pain Index N.	0.0527	0.0435	0.0434	0.0495

offers a conservative alternative characterized by lower annual returns and volatility, albeit with heightened sensitivity to misspecification. Meanwhile, the Maximum Diversification strategy strikes a balance between the two, providing moderate robustness. The optimal strategy choice ultimately hinges on the investor's risk tolerance regarding misspecification and their preferences for expected returns and volatility.

6.3. NORMALIZED RISK MEASURES

In this section, we present the results of the normalization process. Table 4 reports the estimated probabilities of obtaining a misspecified covariance matrix as an input for risk-based portfolio strategies.

These results exhibit consistency and proportionality with the L distances presented in Table 2. As predicted by our model, larger L distances correspond to higher probabilities of incorrect covariance matrix estimation. Equation 11 allows us to compute the probability of accuracy, which serves as the foundation for obtaining normalized risk measures.

The outcomes for the two normalized risk measures across the entire investment universe are summarized in Table 5.

It becomes evident that conventional risk measures are sensitive to the precision

Figure 3: **Conventional and Normalized Risk Measures for each strategy in all investment universe**

of covariance matrix estimation, with slight numerical adjustments arising from the normalization process.

Figure 3 provides a graphical representation that complements Table 5, offering a clearer interpretation of the normalization effects across different investment strategies.

The Sharpe Ratio and Pain Index values, adjusted for the probability of achieving these values with certainty, are represented by black bars, whilst the conventional measures are represented by light blue bars.

As discussed in the previous section, these normalized measures incorporate adjustments that account for the likelihood of accurately estimating the covariance matrix, yielding a refined assessment of risk and performance.

The key insights from the analysis are as follows:

- **Monte Carlo Simulations indicate that the L distance is highest for the $Min - Vol$ strategy across all investment universes**. This elevated distance results in significant adjustments to the normalized Sharpe Ratio and Pain Index, leading to lower final values. The high probability of misspecification directly affects the expected risk-adjusted performance of this strategy.

- **The $Inv-Vol$ and ERC strategies exhibit minimal adjustments**. Both strategies demonstrate robustness to covariance matrix misspecification, as reflected by their low associated L distances. Consequently, the probability of accurate estimation remains high, and the corresponding risk measures experience only minor adjustments.

- **The $Max-Div$ strategy occupies an intermediate position regarding misspecification risk**. Its L distance is moderate, leading to adjustments that are more pronounced than those of the $Inv-Vol$ and ERC strategies but significantly smaller than those observed for the $Min-Vol$ strategy.

From Figure 3, we can infer that investors' strategy selection may depend on their tolerance for estimation risk and their interpretation of the normalized risk measures. The adjustments resulting from the normalization process provide a nuanced understanding of each strategy's robustness and reliability in the presence of covariance matrix misspecification.

7. CONCLUDING REMARKS

Risk-based portfolios are widely regarded as robust allocation methods due to their reliance solely on the covariance matrix of asset returns, thereby circumventing the need to estimate mean returns. This dependency underscores the critical importance of minimizing errors in covariance matrix estimation, as inaccuracies can propagate through allocation decisions, resulting in suboptimal portfolio construction.

Our analysis demonstrates that different investment strategies exhibit varying degrees of sensitivity to model misspecification. Notably, the $Min-Vol$ (Minimum Volatility) and $Max-Div$ (Maximum Diversification) strategies show the highest levels of discrepancy, as quantified by the L distance, when compared to other portfolio strategies.

In response to these findings, we leverage the L distance as a robust metric to evaluate the impact of covariance misspecification across diverse investment environments. To further refine this evaluation, we propose novel risk measures and introduce a normalization process aimed at improving the interpretability and reliability of risk assessments.

The normalization process systematically adjusts the value of risk measures by incorporating the probability of correctly specifying the model. This adjustment either amplifies or attenuates the risk measure based on the likelihood of accurately estimating the covariance matrix. Although the Sharpe Ratio and the Pain Index are used as illustrative examples, the normalization methodology is generalizable to any risk measure employed in risk-based portfolio allocations.

Our results reveal that the $Min-Vol$ portfolio undergoes the most substantial correction in normalized risk measures, highlighting its pronounced sensitivity to covariance matrix misspecification. This underscores the critical role of normalization in mitigating distortions in risk-based strategies arising from input inaccuracies.

In summary, this study offers key insights for portfolio managers and financial practitioners by elucidating the sensitivity of risk-based portfolios to covariance matrix misspecification. The proposed risk measures and normalization techniques provide a structured framework for identifying, assessing, and mitigating risks associated with estimation errors. By refining conventional risk measures to account for model inaccuracies, stakeholders are better equipped to make informed investment decisions, ultimately enhancing the robustness and resilience of portfolio strategies. These contributions are particularly significant in volatile and uncertain market conditions, where precise risk estimation and adjustment are paramount for achieving optimal outcomes.

Despite its contributions, this study is not without limitations. First, the reliance on simulated covariance matrices and assumptions inherent in the Dynamic Conditional Correlation (DCC) model may not fully capture the intricate dynamics of real-world financial markets. Second, the focus on a limited set of risk-based strategies may restrict the generalizability of the findings to other allocation methodologies. Third, practical considerations such as transaction costs and liquidity constraints were not incorporated into the analysis, despite their relevance in portfolio optimization. Lastly, the normalization process, while innovative, relies on the accurate estimation of the probability of model correctness-a challenging task in practical applications. Future research can address the study's limitations by validating findings with empirical datasets and exploring alternative covariance estimation models, such as Dynamic Equicorrelation (DECO, Engle and Kelly (2011)). Expanding the analysis to include additional risk-based strategies and integrating transaction costs and liquidity constraints would enhance real-world applicability. Improved statistical methods, including Bayesian approaches and model-free techniques, could refine the estimation of model correctness probabilities. Stress testing, out-of-sample validation, and scenario analysis under varying market conditions would further strengthen the framework. At the same time, alternative modeling approaches will be considered by analyzing a broader array of risk-based strategies (such as, for example, Kelly portfolios (Carta and Conversano, 2020)), and incorporating real-world constraints such as transaction costs and liquidity factors into the evaluation framework. Additionally, the cost of misspecification error will also be evaluated in ex-post style analysis models (Conversano and Vistocco, 2010). Such efforts will further enhance the applicability and robustness of risk-based portfolio management in diverse financial contexts.

Declarations

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