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BAYESIAN INFERENCE FOR EXPONENTIATED INVERTED WEIBULL DISTRIBUTION IN PRESENCE OF PROGRESSIVE TYPE-II CENSORING

Teena Goyal
Department of Mathematics & Statistics, Banasthali Vidyapith, Rajasthan, India

Piyush K Rai, Mahaveer S Panwar
Department of Statistics, Banaras Hindu University, Varanasi, India
Sandeep K Maurya [1]
Department of Statistics, Central University of South Bihar, Bihar, India

Abstract *The present article gives the point as well as interval estimates for the parameters and lifetime characteristics as reliability and hazard rate functions of the exponentiated inverted Weibull distribution in presence of progressive type-II censored data under classical and Bayesian approach. The point estimates under classical paradigm are obtained with the help of maximum likelihood estimation procedure and in case of Bayesian paradigm, gamma prior is used for both unknown parameters under squared error and linex loss functions. The Metropolis-Hasting algorithm is applied to generate MCMC samples from the posterior density. In case of interval estimation; bootstrap confidence intervals (Boot-t and Boot-p) and highest posterior density intervals for the unknown parameters are computed. The performance of these estimates are studied on the basis of their simulated risks and length of intervals. Additionally, a real dataset is used to illustrate the proposed censoring technique and a simulation study is used to support the given study.*

Keywords: *Progressive censoring, Bayes estimation, Loss function, Metropolis-Hastings algorithm, Simulated risk.*

1. Introduction

In life testing experiments, Weibull distribution is one of the most suitable, applicable and famous model among the other existing lifetime models. As a result of the wide diversity of the versatile mechanism, the two parameter Weibull distribution is used on a large scale in the field of survival and reliability theory; especially for the non-censored data. Generally, there are three parameters involved in the above distribution namely; scale, shape and location parameters. The estimation of these parameters can be possible by using the different methods available in the statistical literature as graphically and analytically. Jiang and

[1] sandeepmaurya.maurya48@gmail.com

Murthy (1999) has introduced an approach to characterize the parameters of exponentiated Weibull distribution graphically. Analytical methods include maximum likelihood estimation, least squares estimation and method of moments, etc. The analytical techniques of estimation of the parameters are more accurate and reliable than the graphical methods (see Dolas et al. (2014)). After that, Mudholkar and Srivastava (1993) has proposed generalization of the Weibull distribution as exponentiated Weibull distribution and studied its statistical properties. Later, Flaih et al. (2012) has proposed a generalization of inverted Weibull (IW) distribution, named as exponentiated inverted Weibull (EIW) distribution, by adding one more shape parameter exponentially in IW distribution.

Let X be a random variable, said to follow EIW distribution if its probability density function (PDF) is given as:

$$f(x;\theta,\beta) = \theta\beta x^{-(\beta+1)}(e^{-x^{-\beta}})^{\theta}; \qquad x>0, \quad \theta>0, \quad \beta>0 \tag{1}$$

here, θ and β both are the shape parameters. If we put $\theta = 1$, it will convert to its baseline inverse Weibull distribution and if we put $\beta = 1$ then, it becomes the exponentiated inverted exponential distribution. The reliability function of the EIW distribution is given by

$$R(t;\theta,\beta) = 1-(e^{-t^{-\beta}})^{\theta}; \qquad t>0, \quad \theta>0, \quad \beta>0$$

and its associated hazard rate is

$$h(t;\theta,\beta) = \frac{\theta\beta t^{-(\beta+1)}(e^{-t^{-\beta}})^{\theta}}{1-(e^{-t^{-\beta}})^{\theta}}; \qquad t>0, \quad \theta>0, \quad \beta>0.$$

Flaih et al. (2012) mentioned that the shape of the PDF of the EIW distribution are uni-model and hazard rate function has upside-down bathtub nature. Singh et al. (2002) has discussed the estimation of parameters for the exponentiated Weibull family under linex loss function and the same distribution is studied by Singh et al. (2005) for the censored data. Kundu and Howlader (2010a) has discussed the inferences and prediction of the inverse Weibull distribution in the case of censored data. Flaih et al. (2012) has discussed the model selection between EIW and IW distributions.Later, Ahmad et al. (2014) has explained the estimation of EIW distribution under asymmetric loss functions.

The case of censored data do arise when a researcher may receive incomplete or partially known data. Specially in reliability and life-testing experiments, in which items are either lost or removed from experiment before its failure, intentionally or unintentionally. For example, in an experiment, if an individual gives

up from the experiment, accidental breakage occurs or some abrupt circumstances arises like unavailability of testing facilities etc. in such cases, the complete sample information are difficult to be found. These type of datasets are know as censored data. In general, there are two conventional censoring schemes named, type-I and type-II censoring schemes.
The type-I censoring scheme, due to time constraint, also known as time censoring. Under this scheme, the number of failure observations is random and may vary from experiment to experiment but the time duration of the study is fixed in advanced for each of the experiment. This censoring scheme has advantage of saving time duration of the experiment. While, in the type-II censoring, investigator fixes the number of observations before the experiment started. Such type censoring is also known as failure censoring, since the number of the observations is prefixed. Here, duration of the life-test is a random variable i.e. it may vary from experiment to experiment whereas, the number of observation is fix known constant, thus, it ensures the availability of a fixed number of observations for the study. See Ng et al. (2006), Balakrishnan et al. (2007), Kundu and Howlader (2010b), Joarder et al. (2011), Dey and Kundu (2012), Han and Kundu (2015), Prajapati et al. (2020) and Goyal et al. (2019) etc. for more details about censoring scheme and estimation of the parameter for censored data. The type-II censoring ensures about the number of observations, it guarantees the desired efficiency of the inferential procedure. But these two censoring schemes do not allow the dropping or removal of any experimental unit before their failure. Later Balakrishnan and Sandhu (1995) has discussed the algorithm of a more advanced censoring scheme called, progressive type-II censoring (PTIIC) scheme, which allows the flexibility of removals. PTIIC is the generalization of failure censoring (type-II) scheme. Initially, the PTIIC scheme has been discussed by Herd (1956), even though he referred to them as “multi-censored samples”. After that, the importance and applicability of the progressive censoring scheme have been discussed by Cohen (1963) and Viveros and Balakrishnan (1994) obtained the interval estimation of lifetime under PTIIC scheme. Later Balakrishnan et al. (2003), Kundu (2008) and Kundu and Biswabrata (2009) have discussed the scheme for different distributions. Almetwally et al. (2023) have discussed the Bayesian analysis under progressive type -II censoring for unit- Weibull distribution. For more literature, reader may refers to Aggarwala and Balakrishnan (1998), Ng and Chan (2007), Raqab et al. (2010) and El-Din and Shafay (2013) etc.

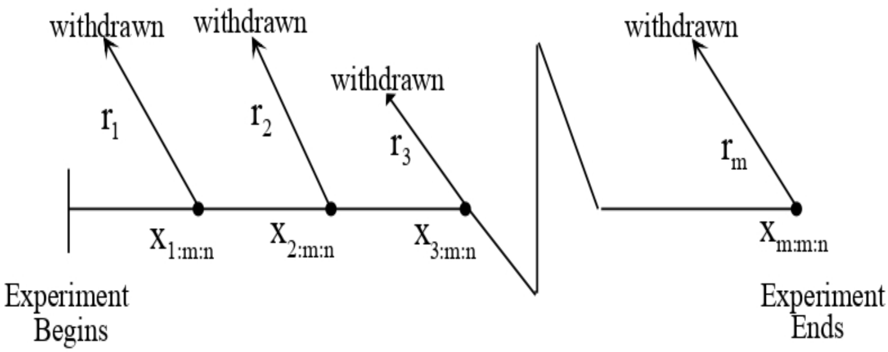

Figure 1: Schematic representation of progressive Type-II censoring scheme.

1.1. Progressive Censoring & its Likelihood Function

In the PTIIC scheme, along with the failure items, removals are also play an important role. The scheme is discussed below.

1. Suppose 'n' units are placed in a life testing experiment at time zero (starting point of time) and 'm' failure times are going to be observed with pre-decided removal scheme $r = (r_1, r_2, ..., r_m)$. Here all $r_i' s$ $(i = 1, 2, \cdots, m)$ are positive integers.

2. At the time of first failure, r_1 of the surviving units randomly selected from the remaining '$n-1$' units and removed. At the time of second failure, r_2 of the surviving units are randomly selected from the resting '$n-2-r_1$' items and removed from the life-test experiment.

3. After that, at the time of m^{th} failure, all the waiting units $r_m = n - r_1 - r_2 - ... - m$ are removed and then the experiment is stopped.

Here, the observed failure times are denoted by $x_{i:m:n}$, where i denotes the i^{th} failure time, m denotes the total number of observation required (prefixed) and n denotes the total number of items placed on life test. Thus, the observed sample information in PTIIC scheme is $x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}$. A pictorial representation of PTIIC scheme is given in Figure 1.

The likelihood function in the PTIIC scheme, (see Balakrishnan and Sandhu (1995)) is as follows;

$$L(x_i \mid \theta, \beta) = C \prod_{i=1}^{m} f(x_i \mid \theta, \beta)[1 - F(x_i \mid \theta, \beta)]^{r_i} \tag{2}$$

where C is the constant, formulated as:

$$C = n \times (n - r_1 - 1) \times \cdots \times (n - r_1 - r_2 - \cdots - r_m - 1 - m + 1). \tag{3}$$

The main emphasis of this paper is to test the efficacy of Bayes estimates for the parameters of the EIW distribution based on PTIIC. Motivated by these literature, here, we are trying to find better estimator for the parameter of EIW distribution using squared error and linex loss functions. The rest of the article is organized as follows: Section 2, deals with the classical estimation of parameters and Section 3, discusses the technique of Bayesian estimation for the parameters with very short description of prior, loss functions, posteriors, and M-H algorithm. Algorithm for a generation of the sample from PTIIC scheme is introduced in Section 4. In Section 5, techniques of parametric bootstrap confidence interval and HPD interval are discussed to construct the CIs for the unknown parameters. A brief study on simulation is discussed in Section 6. Particular real dataset is analyzed in the Section 7, and the conclusions of the present paper are commented in the last Section 8.

2. Classical Estimation

In this section, we have discussed the MLE of the parameters θ and β based on the data observed under the PTIIC scheme (as discussed in the Subsection 1.1).

2.1. Maximum Likelihood Estimator

Let $x = \{x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}\}$ be a random sample from the EIW distribution with the PDF given in the equation (1). Then, the likelihood function by using the equation (2) is

$$l(\theta, \beta \mid x) = C \prod_{i=1}^{m} \theta \beta {x_i}^{-(\beta+1)} (e^{-{x_i}^{-\beta}})^{\theta} [1 - (e^{-{x_i}^{-\beta}})^{\theta}]^{r_i}.$$

And, the log of likelihood function is

$$L(\theta, \beta \mid x) = \log C + \sum_{i=1}^{m} \log \left[\theta \beta {x_i}^{-(\beta+1)} (e^{-{x_i}^{-\beta}})^{\theta} [1 - (e^{-{x_i}^{-\beta}})^{\theta}]^{r_i} \right]. \tag{4}$$

Where constant C is given in equation (3). Therefore, the ML estimator of the parameter θ and β can be obtained by differentiating the log likelihood function with respect to the corresponding parameters and equating to zero, we get

$$\frac{\partial \log L}{\partial \theta} = \sum_{i=1}^{n} \left[\frac{1}{\theta} - {x_i}^{-\beta} + r_i \frac{(e^{-{x_i}^{-\beta}})^{\theta} {x_i}^{-\beta}}{\{1 - (e^{-{x_i}^{-\beta}})^{\theta}\}} \right] = 0.$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log x_i + \theta \sum_{i=1}^{n} \frac{\log x_i}{x_i^{\beta}} - \sum_{i=1}^{n} r_i \frac{(e^{-x_i^{-\beta}})^{\theta} \theta x_i^{-\beta} \log x_i}{\{1-(e^{-x_i^{-\beta}})^{\theta}\}} = 0.$$

Since the above likelihood equations are not in close form and thus can not be solved it analytically. Therefore, to obtain the the solution from these likelihood equations, we have used Newton-Raphson method based on an iterative procedure. After getting the ML estimator of the parameters, the estimated reliability $\hat{R}_{ML}$ and hazard rate function $\hat{h}_{ML}$ at specific time t can be obtained by using the in-variance property of ML estimator. Thus,

$$\hat{R}_{ML} = 1-(e^{-t^{-\hat{\beta}_{ML}}})^{\hat{\theta}_{ML}}; \qquad t>0, \quad \theta>0, \quad \beta>0.$$

And

$$\hat{h}_{ML} = \frac{\hat{\theta}_{ML}\hat{\beta}_{ML} t^{-(\hat{\beta}_{ML}+1)} (e^{-t^{(-\hat{\beta}_{ML})}})^{\hat{\theta}_{ML}}}{1-(e^{-t^{-\hat{\beta}_{ML}}})^{\hat{\theta}_{ML}}}; \qquad t>0, \quad \theta>0, \quad \beta>0.$$

Here, $\hat{\theta}_{ML}$ and $\hat{\beta}_{ML}$ are the ML estimator of the parameters θ and β respectively.

3. Bayes Method of Estimation

In this section we have considered another approach namely Bayesian method. The reason behind the consideration is that, in the Bayesian approach added more flexible and accurate result as it incorporates prior knowledge with the sample in-formation. Another advantage of the consideration is that the Bayesian approach provides more appropriate results in small as well in large sample.

In Bayesian paradigm, posterior distribution is an effect of two components namely, a prior distribution and the likelihood function, calculated from the statistical model for the observed data. The prior distribution is the distribution of the parameter assumed before the data observed. The choice of the prior distribution may not be easily determined. For the selection of the prior distribution, one can see Berger and Sun (1993), Raqab and Madi (2005) and Singh et al. (2016). There are mainly two different categorization to the prior distribution of parameters defined as proper and improper prior. Another way to defined priors are based on information available in advance and called as informative and non-informative prior. Here, we use the prior distribution for θ and β as $Gamma(a,b)$ and $Gamma(c,d)$ respectively to obtain the posterior distribution.

The choice of the hyper-parameters of the priors (θ, β) are based on the information available in term of prior mean and prior variance. This chosen value of the hyper-parameter may be taken in a way that if we choose any two independent information as prior mean and variance of the priors (θ, β), then $(\mu_1 = a/b, \sigma_1 = a/b^2)$ and $(\mu_2 = c/d, \sigma_2 = c/d^2)$ respectively. Here, μ_1 and μ_2 are the true mean value of the parameter (θ, β) respectively and σ_1 and σ_2 are the true variance of the parameter (θ, β) respectively. Now by using this information, the hyper parameters can be easily evaluated from this relation, $(a = \mu_1/\sigma_1, b = \mu_1^2/\sigma_1)$ and $(c = \mu_2/\sigma_2, d = \mu_2^2/\sigma_2)$ respectively. See Kundu (2008), Singh et al. (2013), Dey et al. (2016), Singh et al. (2016) and El-Sherpieny et al. (2022) for more details about the choice of the hyper-parameters. Now, the joint prior distribution of θ and β is given as

$$\pi(\theta, \beta) \propto \theta^{a-1}\beta^{c-1}e^{-b\theta - d\beta}; \qquad (\theta, \beta) > 0, (a, b, c, d) > 0. \tag{5}$$

3.1. Loss Function

"A loss function is a function that maps an event or the values of one or more variables on a real number intuitively representing some cost associated with the even". In Bayesian statistics, a loss function is used for the estimation of parameters. Here we have considered two different widely used loss functions namely Squared Error Loss Function (SELF) and Linex Loss Function (LLF).

1. *Squared Error Loss Function:* It is a commonly used symmetric loss function, defined as
$$L(\hat{\theta}_{BS}, \theta) = (\hat{\theta}_{BS} - \theta)^2$$
where $\hat{\theta}_{BS}$ is the Bayes estimator under SELF for the given parameter θ.

2. *Linex Loss Function:* This loss function is an asymmetric loss function. Zellner (1986) proposed this loss function for the estimation and prediction of a scaler parameter. The form of LLF is given as
$$L(\hat{\theta}_{BL}, \theta) = e^{\delta(\hat{\theta}_{BL} - \theta)} - \delta(\hat{\theta}_{BL} - \theta) - 1$$
where $\delta \neq 0$ is a constant which determines the shape of the loss function. In particular, the LLF increases almost linearly for negative error and almost exponentially for positive error. Thus, under this loss function, over estimation is considered to be more serious than the under estimation. The behavior of the LLF for the small values of δ, is approximately same as the SELF.

3.2. Posterior Probability Density Function

Let $x = \{x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}\}$ be a random sample from EIW distribution and the parameters θ and β have prior probabilities $\pi(\theta)$ and $\pi(\beta)$ respectively. Then by using equations (4) and (5), the joint posterior density function is given as:

$$\begin{aligned}\pi(\theta,\beta \mid x) &= \frac{\pi(\theta,\beta)L(\theta,\beta \mid x)}{\int_0^\infty\int_0^\infty \pi(\theta,\beta)L(\theta,\beta \mid x)d\theta d\beta}; \qquad \theta>0, \beta>0 \\ &\propto \theta^{a-1}\beta^{c-1}e^{-b\theta-d\beta}\prod_{i=1}^{m}\theta\beta {x_i}^{-(\beta+1)} \\ &\times (e^{-{x_i}^{-\beta}})^{\theta}[1-(e^{-{x_i}^{-\beta}})^{\theta}]^{r_i}. \end{aligned} \tag{6}$$

Now, the Marginal posterior densities of the parameters θ and β can be obtained by integrating the equation (6) with respect to β and θ respectively. And it can be written as equation (7) and equation (8) respectively as

$$\begin{aligned}\pi(\theta \mid x_i) &= \int_0^\infty \pi(\theta,\beta \mid x_i)d\beta \\ &\propto \int_0^\infty \theta^{a-1}\beta^{c-1}e^{-b\theta-d\beta}\prod_{i=1}^{m}\theta\beta {x_i}^{-(\beta+1)} \\ &\times (e^{-{x_i}^{-\beta}})^{\theta}[1-(e^{-{x_i}^{-\beta}})^{\theta}]^{r_i}d\beta. \end{aligned} \tag{7}$$

$$\begin{aligned}\pi(\beta \mid x_i) &= \int_0^\infty \pi(\theta,\beta \mid x_i)d\theta \\ &\propto \int_0^\infty \theta^{a-1}\beta^{c-1}e^{-b\theta-d\beta}\prod_{i=1}^{m}\theta\beta {x_i}^{-(\beta+1)} \\ &\times (e^{-{x_i}^{-\beta}})^{\theta}[1-(e^{-{x_i}^{-\beta}})^{\theta}]^{r_i}d\theta. \end{aligned} \tag{8}$$

3.2.1. Bayes Estimator under SELF

The Bayes estimator under the SELF is nothing but the posterior mean of the corresponding parameters. Let it is denoted by $\hat{\theta}_{BS}$ and. Therefore, the Bayes estimator of the parameter θ can be obtained as:

$$\begin{aligned}
\hat{\theta}_{BS} &= E(\hat{\theta}) = \int_0^\infty \theta\pi(\theta \mid x_i)d\theta \\
&\propto \int_0^\infty \theta \int_{i=0}^{\infty} \theta^{a-1}\beta^{c-1}e^{-b\theta-d\beta}C\prod_{i=1}^{m}\theta\beta \\
&\times \; {x_i}^{-(\beta+1)}(e^{-{x_i}^{-\beta}})^{\theta}\left[1-(e^{-{x_i}^{-\beta}})^{\theta}\right]^{r_i} d\beta d\theta.
\end{aligned}$$

Similarly, the Bayes estimator of the parameter β can be obtained as:

$$\begin{aligned}
\hat{\beta}_{BS} &= E(\hat{\beta}) = \int_0^\infty \beta\pi(\beta \mid x_i)d\beta \\
&\propto \int_0^{\infty} \beta \int_0^{\infty} \theta^{a-1}\beta^{c-1}e^{-b\theta-d\beta}C\prod_{i=1}^{m}\theta\beta \\
&\times \; {x_i}^{-(\beta+1)}(e^{-{x_i}^{-\beta}})^{\theta}\left[1-(e^{-{x_i}^{-\beta}})^{\theta}\right]^{r_i} d\theta d\beta.
\end{aligned}$$

3.2.2. Bayes Estimator under LLF

Since, the Bayes estimator for the parameter θ under LLF (say $\hat{\theta}_{BL}$) is defined as $\left[\frac{-1}{a}\log E(e^{-a\theta})\right]$. Thus, for the considered distribution, the expression is given as:

$$\begin{aligned}
\hat{\theta}_{BL} &= \frac{-1}{a}\log\left[\int_0^{\infty} e^{-a\theta}\pi(\theta \mid x_i)d\theta\right] \\
&\propto \frac{-1}{a}\log\int_0^{\infty} e^{-a\theta}\int_0^{\infty} \theta^{a-1}\beta^{c-1}e^{-b\theta-d\beta} \\
&\times \prod_{i=1}^{m}\theta\beta{x_i}^{-(\beta+1)}(e^{-{x_i}^{-\beta}})^{\theta}\left[1-(e^{-{x_i}^{-\beta}})^{\theta}\right]^{r_i} d\beta d\theta.
\end{aligned}$$

Similarly, the Bayes estimator for the parameter β is:

$$\begin{aligned}\hat{\beta}_{BL} &= \frac{-1}{a}\log\left[\int_0^\infty e^{-a\beta}\pi(\beta\mid x_i)d\beta\right]\\ &\propto \frac{-1}{a}\log\int_0^\infty e^{-a\beta}\int_0^\infty \theta^{a-1}\beta^{c-1}e^{-b\theta-d\beta}\\ &\times \prod_{i=1}^{m}\theta\beta x_i^{-(\beta+1)}(e^{-x_i^{-\beta}})^{\theta}[1-(e^{-x_i^{-\beta}})^{\theta}]^{r_i}d\theta d\beta.\end{aligned}$$

3.3. MCMC Simulation

The expressions for the Bayes estimators under SELF and LLF are not in closed form. So it needed some algorithm to draw sample and find its estimates. One of the famous and efficient technique of doing so is Markov Chain Monte Carlo (MCMC) method. One may refer Gilks et al. (1995), Gelfand (1996), Dagpunar (2007), Marin and Robert (2007), Chen et al. (2012) and Robert and Casella (2013) for more about MCMC technique. The integral involved in Bayes estimators do not solve analytically. In such a situation, MCMC methods namely Metropolis-Hastings (M-H) algorithm (see Hastings (1970)) can be effectively used.

To obtain the MCMC samples from the posterior probability $\pi(\theta \mid data)$, using the Metropolis-Hastings (M-H) algorithm, we have considered a normal distribution as the proposal density i.e. $N(\mu,\Sigma)$ where Σ is the variance-covariance matrix. It may be the point here that, if we generate observation from the normal distribution, we may get negative values also which are not possible as the parameters under consideration are positive valued. Therefore, we take the absolute value of generated observation. The M-H algorithm starts with an initial value of the parameter say θ^0 and specified a rule for simulating the t^{th} value in the sequence θ^t given the $(t-1)^{st}$ value in the sequence θ^{t-1}. This rule consists of a proposal density which simulates a candidate value say θ^* and acceptance probability say P. This algorithm can be described as follows:

1. Set the initial guess of parameter θ, say θ^0 from uniform $U(0,1)$.
2. Simulate a candidate value θ^* from a proposal density $p(\theta^* \mid \theta^{t-1})$.
3. Compute the ratio R= $\frac{\pi(\theta^*)p(\theta^{t-1}|\theta^*)}{\pi(\theta^{t-1})p(\theta^*|\theta^{t-1})}$.
4. Compute acceptance probability $P = \min\{R,1\}$.

5. Take a sample value θ^t, such that, $\theta^t = \theta^*$ with probability P; otherwise $\theta^t = \theta^{t-1}$.

After getting MCMC samples from posterior distribution, we can find the Bayes estimates for the parameters in the following way

$$E(\theta|data) = \frac{1}{N-N_0}\sum_{i=N_0+1}^{N}\theta_i$$

where N_0 is burn-in period of Markov chain and N be the sufficiently large number of replications. In using the above algorithm, the problem arises how to choose the initial guess. Here, we propose to the use of ML estimate of the parameter θ, obtained by using the method described in subsection 2.1, as the initial value for MCMC processes. The choice of covariance matrix Σ is also an important issue, one can follow Ntzoufras (2011) for more details. One choice for Σ would be the asymptotic variance-covariance matrix $I^{-1}(\hat{\theta})$. While generating M-H samples by taking $\Sigma = I^{-1}(\hat{\theta})$, we noted that the acceptance rate for such a choice of Σ is about 30%. By acceptance rate, we mean the proportion of times a new set of values is generated at the iteration stages. It is well known that if the acceptance rate is low, a good strategy is to run a small pilot run using diagonal Σ as a rough estimate of the correlation structure for the target posterior distribution and then re-run the algorithm using the corresponding estimated variance-covariance matrix; for more detail see (Gelman et al., 2013, pp. 334-335), Kaushik et al. (2017) and Maurya et al. (2017).

4. Algorithm for Sample Generation under PTIIC Scheme

We have used the following steps to generate a PTIIC sample from the EIW distribution. The steps are:

1. Specify the values of n, m, $\underset{\sim}{\alpha} = (\theta,\beta)$ and $r = (r_1, r_2, \ldots, r_m)$.

2. Generate m i.i.d. random numbers $w_1, w_2, \ldots, w_m$ from uniform $U(0,1)$ distribution.

3. Set $V_i = w_i^{1/(i+\sum_{j=m-i+1}^{m} r_j)}$; for $i = 1,2,\ldots,m$.

4. Set $U_{i:m:n} = 1 - V_m V_{m-1} \ldots V_{m-i+1}$ for $i = 1,2,\ldots,m$. Then $U_{1:m:n}, U_{2:m:n}, \ldots$, $U_{m:m:n}$ are the required PTIIC sample from the uniform $U(0,1)$ distribution.

5. Finally, set $x_{i:m:n} = F^{-1}(U_{i:m:n})$, for $i = 1,2,\ldots,m$, where $F^{-1}(\cdot)$ is the inverse distribution function of EIW distribution.

Then, $x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}$ are the required n random PTIIC sample from the EIW distribution.

5. Interval Estimation

In this section we have computed confidence intervals for the parameters of the EIW distribution under classical and the Bayesian setup. In classical setup, we have calculated parametric boot strep intervals namely; Boot-p and Boot-t. While in the Bayesian setup, we have calculated the highest posterior density (HPD) intervals. The details about these intervals are given below.

5.1. Bootstrap Confidence Interval

Sometime the class intervals based on the asymptotic property or the normal theory assumption do not perform good for small samples. In that situation, the use of bootstrap methods, one can obtain the accurate intervals without using the normal theory assumption. The bootstrap methods make computer-based adjustments to the standard intervals endpoints and surely improve the coverage accuracy by an order of magnitude, at least asymptotically. Here, we have discussed two types of CIs using bootstrap method. The parametric percentile bootstrap (Boot-p) suggested by Efron (1982) and parametric studentized bootstrap (Boot-t), suggested by Hall (1988). See Efron (1992) and DiCiccio and Efron (1996) for more details about bootstrap CIs.

5.1.1. Parametric Boot-p

An algorithm for the Boot-p CIs is as follows:

1. Assemble the PTIIC data and obtain ML estimators for the parameters θ & β, denoted as $\hat{\theta}_{ML}$ & $\hat{\beta}_{ML}$.
2. Generate a PTIIC sample by using ML estimators of the parameters based on pre-specified removal scheme $r = (r_1, r_2, \ldots, r_m)$.
3. Generate B number of bootstrap samples from the above generated samples.
4. Obtain ML estimators for each B bootstrap sample, denoted as $\left\{\hat{\theta}_1^*, \hat{\beta}_1^*\right\}$, $\left\{\hat{\theta}_2^*, \hat{\beta}_2^*\right\}$, ..., $\left\{\hat{\theta}_B^*, \hat{\beta}_B^*\right\}$.
5. Arrange these generated samples in ascending orders as $\left\{\hat{\theta}_{(1)}^*, \hat{\theta}_{(2)}^*, \ldots, \hat{\theta}_{(B)}^*\right\}$ and $\left\{\hat{\beta}_{(1)}^*, \hat{\beta}_{(2)}^*, \ldots, \hat{\beta}_{(B)}^*\right\}$.

A pair of $100(1-\alpha)\%$ Boot-p CIs for θ & β are given by $\left[\hat{\theta}^*_{(B\alpha/2)}, \hat{\theta}^*_{(B(1-\alpha/2))}\right]$ and $\left[\hat{\beta}^*_{(B\alpha/2)}, \hat{\beta}^*_{(B(1-\alpha/2))}\right]$ respectively.

5.1.2. Parametric Boot-t

The algorithm for generating p-Boot is very simple, though, if the sample size is very small then, percentile approach is not so much accurate. Thus, in this condition, studentized t bootstrap (Boot-t) approach can be used. It gives more accuracy to results than the percentile approach. The algorithm of the Boot-t CIs is just an extension of the algorithm of p-Boot.

5. Repeat step 1-4 as in Boot-p approach.

6. Compute, standard errors of the parameters, denoted as $\{\hat{se_1}^*(\theta), \hat{se_1}^*(\beta)\}$, $\{\hat{se_2}^*(\theta), \hat{se_2}^*(\beta)\}$, $\cdots$, $\{\hat{se_B}^*(\theta), \hat{se_B}^*(\beta)\}$.

7. Compute, statistics $z^*_B(\theta) = \frac{\hat{\theta}^*_B - \hat{\theta}_{ML}}{\hat{se_B}^*(\theta)}$ and $z^*_B(\beta) = \frac{\hat{\beta}^*_B - \hat{\beta}_{ML}}{\hat{se_B}^*(\beta)}$.

8. Arrange $z^*_B(\theta)$ and $z^*_B(\beta)$ in ascending orders, denoted as $z^*_{(B)}(\theta)$ and $z^*_{(B)}(\beta)$. A pair of $100(1-\alpha)\%$ Boot-t CIs for θ & β are given by

$$\left[\hat{\theta}_{ML} - z^*_{(B(1-\alpha/2))}(\theta) * \hat{se}(\theta),\ \hat{\theta}_{ML} + z^*_{(B\alpha/2)}(\theta) * \hat{se}(\theta)\right]$$

and

$$\left[\hat{\beta}_{ML} - z^*_{(B(1-\alpha/2))}(\beta) * \hat{se}(\beta),\ \hat{\beta}_{ML} + z^*_{(B\alpha/2)}(\beta) * \hat{se}(\beta)\right]$$

respectively.

5.2. Highest Posterior Density Interval

The HPD credible intervals (see Box and Tiao (1973) and Chen and Shao (1999)) of the parameter $\underset{\sim}{\alpha} = (\theta, \beta)$ are obtained on the basis of ordered MCMC samples of α as $\underset{\sim}{\alpha}_{(1)}, \underset{\sim}{\alpha}_{(2)}, \cdots, \underset{\sim}{\alpha}_{(N)}$. After that, $100(1-\alpha)\%$ credible interval for the parameter $\underset{\sim}{\alpha}$ is obtained as $((\underset{\sim}{\alpha}_{(1)}, \underset{\sim}{\alpha}_{[(1-\alpha)N]+1}), \cdots, (\underset{\sim}{\alpha}_{[N\alpha]}, \underset{\sim}{\alpha}_N).)$ Where $[Y]$ denotes the largest integer less than or equal to Y. The, HPD credible intervals provides shortest length CIs. See also Edwards et al. (1963), Ng et al. (2006), Kundu and Howlader (2010b) and Singh et al. (2013) etc.

6. Simulation Study

In this section we have performed simulation study based on the PTIIC samples for EIW distribution and estimates the parameters of the model under the above discussed classical and Bayesian methods. We have also calculated the CIs for the model parameters along with the model survival and hazard rate functions. We have calculated here the simulated risk function also. Performance of Bayes estimators and ML estimators are examined based on the simulation. The steps involve to perform the study are enumerated as follows.

1. Generate PTIIC samples using the algorithm discussed in the Section 3 for particular values of n, m, θ, β and r.

2. The ML estimators of the parameters, reliability and hazard rate function have been computed for these particular values.

3. In case of Bayesian analysis, we have assumed that both the parameters have gamma prior. The chosen values of the hyper-parameters are taken as $a = 0.2$, $b = 0.2$, $c = 0.2$ & $d = 0.2$, as particular case. The reason behind the choice of this hyper-parameter is to the consideration of informative prior. Also, the choice of hyper-parameters for the gamma prior should be guided by a combination of prior knowledge. One can choose large variance prior in case of lack of prior knowledge or it may be appropriate to choose hyper parameters that result in relatively flat or non-informative priors. As, we know that, in EIW distribution, both of the parameters play the role of shape parameters and both are sensitive for the shape of the distribution. So here, we have chosen same combination for choice of this hyper-parameter, as true mean 1 and true variance 5 for both of the hyper-parameters. Also, for the considered combination, when mean>variance, both the gamma priors of the parameters covers wide variation. For more details, see Section 3.

4. M-H algorithm of MCMC technique has been used to generate posterior samples.

5. From these simulated posterior samples, the Bayes estimators of the parameters, survival and hazard rate under the assumption of the above prior using SELF and LLF have been obtained.

6. Only one choice of loss function parameter δ is considered ($\delta = 0.1$ as particular case) for LLF.

7. Boot-p and Boot-t CIs have obtained under classical set-up. And in Bayesian paradigm, we have constructed 95% HPD CI for both the parameters.

8. The values of the estimates, survival and hazard rate at time $t = 1$ have been reported in the Tables 5-7. Here the arbitrary chosen true value of the parameters (θ, β) are taken as $(2, 2)$. Table 8, shows the risks of these parameters under different estimation techniques and CIs for these estimators are mentioned in Tables 9-10.

9. The estimates of the parameters θ & β for $m = 32$ for varying combinations of values of the parameters as (θ, β) = (1.2, 2), (3, 2), (2, 0.5) & (2, 1.5) with their associated risks and estimators of survival and hazard rate functions in presence of different removal schemes under PTIIC as R_1, R_2, R_3 & R_4 have been tabulated from Tables 11-13.

The Figure 2 gives an idea about the quantiles of the parameters θ and β respectively. These are based on the MCMC samples, which explains the probability $(P(X \leq x) > 0.1, 0.5, 0.9)$, where X is a random variable. In our case, it is for θ and β parameters and x is the particular values of the MCMC sample.

For simulation study, we have generated different random samples of size n = 50, m = 20, 30, 32 and 40 with the parameters $\theta = 2$ and $\beta = 2$ from EIW PTIIC and taken different censoring schemes R_1,R_2,R_3 and R_4. The complete schemes along with these parameters values are given in Table 4. The simulation results for these schemes are given in the Tables 5-13.

Under different censoring schemes (see Table 4) p^*q (means number p is repeated q times) and for different parameter values, we conclude the following:

1. From the Table 8, the risks of the Bayes estimators are lesser as compared to the risk of the ML estimators. This table also shows that the risk of the ML estimator for the parameter θ is less than the parameter β for all the considered value of m and considered different removable schemes. But no such pattern found in case of risk of the Bayes estimators under different loss functions except $m = 20$, which shows the reverse result as in classical ML estimation under different removable schemes.

2. From the Table 8, we observed that the risk of the Bayes estimators of the parameters, under LLF, is consistently smaller than the risk of the estimators under SELF.

3. This table also shows that, for all the considered values of m, the Bayes risk under LLF are concentrated and converses to zero. So, this method may be accepted for this distribution.

4. From Tables 9 and 10, we see that the length of HPD intervals is smaller than the length of Boot-t CI and Boot-t is smaller than Boot-p CI for both the parameters θ and β.

5. From Table 12, we may conclude that the Bayes estimators have lesser risk as compared to classical estimation for different combinations of the parameters θ and β. One point is also noticeable that, for the EIW distribution, both the parameters (θ, β) are play the role of shape parameter and both are sensitive.

6. This table also shows that, for any parameter (θ or β) smaller risk is associated with smaller value of parameter and vice-versa.

7. This table also shows that the under R_2 scheme, for all the combination of the parameters, risk under classical estimators are maximum for both the parameters.

8. This table also shows that, for all the considered parameters (θ or β) the Bayes risks under LLF are concentrated and converses to zero. So, this method may be accepted for this distribution.

7. Real Data Analysis

Here we have considered a real dataset of the remission time (in months) of 128 bladder cancer patients data, to show the applicability of the considered model in classical as well as Bayesian context in complete as well as in censored case. The dataset was reported by Lee and Wang (2003), and is given in Table 1.

The ML estimate of the parameters (θ and β), survival and hazard rate function based on the complete sample ($n = 128$) are obtained as $\hat{\theta}_{ML} = 2.4262$, $\hat{\beta}_{ML} = 0.7551$, $\hat{S}_{ML}(t = 1) = 0.9116$ & $\hat{h}_{ML}(t = 1) = 0.1776$ respectively. The Bayes estimate for the parameters θ and β, survival and hazard rate under SELF are $\hat{\theta}_{BS} = 2.1961$, $\hat{\beta}_{BS} = 0.7370$, $\hat{S}_{BS}(t = 1) = 0.8888$, $\hat{h}_{BS}(t = 1) = 0.2026$ respectively and under LLF estimates are $\hat{\theta}_{BL} = 2.1941$, $\hat{\beta}_{BL} = 0.7370$, $\hat{S}_{BL}(t = 1) = 0.8885$, $\hat{h}_{BL}(t = 1) = 0.2028$ respectively.

A PTIIC sample of size $m = 80$ is selected randomly from the complete sample of size $n = 128$ with the censoring scheme $R = (0^*32, 3^*16, 0^*32)$. The point and the interval estimates of the parameters are given in the Table 2. This table, shows the ML estimates and the Bayesian estimates based on MCMC technique using SELF and LLF for both parameters θ & β along with the interval estimates of both the parameters using bootstrap confidence intervals (Boot-p and Boot-t) technique and highest posterior density intervals.

The point and the interval estimates of the survival and hazard rate function at different times $t = 3, 6$ and 12 are given in Table 3. From this table, one can observed that the length of the interval become shorter with respect to increase in time for all the considered times. And also, the length of the intervals are shortest in case of HPD intervals (here LL stands for lower limit and UL stands for upper limit of the intervals).

8. Conclusion

In this paper, we have proposed point as well as interval estimation under classical and Bayesian context of the parameters for the exponentiated inverse Weibull distribution. We have also estimated the survival and hazard rate functions of the considered model under progressive type-II censoring using maximum likelihood estimation and Bayesian analysis using gamma prior under SELF and LLF. Both the classical and Bayesian analyses methods have their own advantages and limitations, and this depends on factors such as the availability of prior information, the desire for uncertainty quantification. We have also obtained the confidence intervals for the parameters using parametric bootstrap methods (namely Boot-t and Boot-p) and Bayesian HPD intervals. We have taken a simulation study by using MCMC technique to compute the point estimations and their corresponding confidence intervals. From a simulated study, we can conclude that the Bayes estimators with an informative gamma prior may be used particu-larly when prior information is known. We also find that the Bayes risks is always smaller than the classical risks. Also, the Bayes risks under LLF for all the consid-ered parameters (θ or β) values and m more concentrated and converses to zero. So, one can also this method while dealing with EIW distribution. The perfor-mance of HPD intervals seems comparatively good because the computed risks are considerably smaller as compared to classical method as well as the length of the HPD intervals are minimum with respect to the bootstrap intervals.

In the simulation study, the results of estimated parameters and their confidence intervals as Boot-p, Boot-t and HPD intervals have also provided. The characteristics based on the functional form of parameters like survival and hazard rate functions with their confidence intervals have also provided under different censoring schemes.

The results obtained under this study may be motivate to researchers of the field of statistical inference to consider the facts for the better application of EIW distribution in real life testing circumstances.

Figure 2: Cumulative quantile plot for the parameters θ and β.

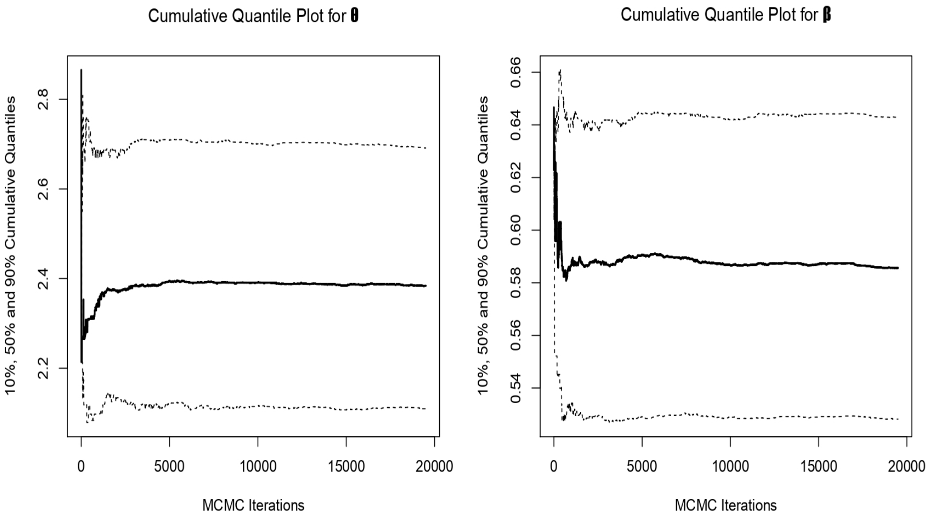

Table 1: Remission times (in months) of 128 bladder cancer patients.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.2	2.23
3.52	4.98	6.97	9.02	13.29	0.4	2.26	3.57	5.06	7.09
9.22	13.8	25.74	0.5	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.7	5.17	7.28	9.74	14.76	6.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
7.39	10.34	14.83	34.26	0.9	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
5.49	7.66	11.25	17.14	79.0	51.35	2.87	5.62	7.87	11.64
17.36	1.4	3.02	4.34	5.71	7.93	11.79	18.1	1.46	4.4
5.85	8.26	11.98	19.13	1.76	3.25	4.5	6.25	8.37	12.02
2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69.		

Table 2: Point and interval estimates of the parameters (θ, β) under different techniques for the real dataset.

	ML	Boot-t	Boot-p	SELF	LLF	HPD
θ	2.7130	(1.7675, 3.5487)	(2.0886, 4.7906)	2.4000	2.3974	(1.9633, 2.8526)
β	0.6134	(0.4326, 0.7373)	(0.5079, 0.893)	0.5858	0.5857	(0.5004, 0.672)

Table 3: Point and interval estimates of survival and hazard rate functions under different techniques and time points for the real dataset.

	ML	Boot-t	Boot-p	SELF	LLF	HPD	ML	Boot-t	Boot-p	SELF	LLF	HPD
time	$\hat{S}(t)$	length	length	$\tilde{S}(t)$	$\tilde{S^*}(t)$	length	$\hat{h}(t)$	length	length	$\tilde{h}(t)$	$\tilde{h^*}(t)$	length
t=3	0.2509	0.127	0.1366	0.2834	0.2837	0.0679	0.2828	0.2295	0.3322	0.2462	0.2459	0.1178
t=6	0.4050	0.0551	0.0513	0.4316	0.4320	0.0245	0.0924	0.0577	0.0728	0.0820	0.0819	0.0295
t=12	0.5539	0.0196	0.0404	0.5713	0.5716	0.017	0.0302	0.0132	0.0137	0.0273	0.0273	0.0066

Table 4: Different censoring schemes (CS) considered for simulation study.

n	m	R_1	R_2	R_3	R_4
	20	0*5,2*15	0*5,4*7,2*1,0*7	0*1,6*2,0*2,2*5,4*2,0*8	0*1,6*2,0*2,2*4,0*6,2*5
50	30	0*20,2*10	0*10,2*10,0*10	4*5,0*25	2*5.0*10,2*5
	32	2*8, 0*23, 2*1	1*18, 0*14	3*6, 0*26	0*23, 1*3, 4*3, 1*3
	40	0*35,2*5	1*10,0*30	2*5,0*35	0*10,2*5,0*25

Table 5: Simulation study of estimated values of θ for varying sample sizes under different estimation methods

		R_1	R_2	R_3	R_4
m=20	$\hat{\theta}_{ML}$	2.8684	3.3939	3.1899	2.692
	$\hat{\theta}_{BS}$	1.8727	2.0429	1.9924	1.8326
	$\hat{\theta}_{BL}$	1.8683	2.0375	1.9871	1.8284
m=30	$\hat{\theta}_{ML}$	2.3942	2.827	2.4023	2.3008
	$\hat{\theta}_{BS}$	1.7508	1.9373	1.7648	1.7136
	$\hat{\theta}_{BL}$	1.7472	1.9328	1.761	1.7101
m=32	$\hat{\theta}_{ML}$	2.3779	2.5968	2.3734	2.3533
	$\hat{\theta}_{BS}$	1.761	1.8608	1.7586	1.7371
	$\hat{\theta}_{BL}$	1.7573	1.8567	1.7548	1.7336
m=40	$\hat{\theta}_{ML}$	2.1382	2.2624	2.2165	2.3674
	$\hat{\theta}_{BS}$	1.6555	1.7226	1.6972	1.7817
	$\hat{\theta}_{BL}$	1.6525	1.7193	1.6939	1.7781

Table 6: Simulation study of estimated values of β for varying sample sizes under different estimation methods

		R_1	R_2	R_3	R_4
m=20	$\hat{\beta}_{ML}$	2.7385	2.5188	2.5785	2.6505
	$\hat{\beta}_{BS}$	1.6956	1.5711	1.6714	1.7423
	$\hat{\beta}_{BL}$	1.6902	1.5666	1.6666	1.7369
m=30	$\hat{\beta}_{ML}$	2.4031	2.4664	2.3547	2.3168
	$\hat{\beta}_{BS}$	1.7898	1.8201	1.831	1.7793
	$\hat{\beta}_{BL}$	1.7861	1.8164	1.8273	1.7757
m=32	$\hat{\beta}_{ML}$	2.3483	2.4667	2.3656	2.3701
	$\hat{\beta}_{BS}$	1.8391	1.8957	1.8581	1.8
	$\hat{\beta}_{BL}$	1.8357	1.8921	1.8546	1.7965
m=40	$\hat{\beta}_{ML}$	2.159	2.2993	2.214	2.3785
	$\hat{\beta}_{BS}$	1.7642	1.879	1.8202	1.9276
	$\hat{\beta}_{BL}$	1.7615	1.8761	1.8174	1.9246

Table 7: Estimated values of survival and hazard rate functions at specific time $t = 1$ when true values of parameters considered $\theta = 2$ and $\beta = 2$

m	RS	$\hat{R}_{ML}$	$\hat{R}_{BS}$	$\hat{R}_{BL}$	$\hat{h}_{ML}$	$\hat{h}_{BS}$	$\hat{h}_{BL}$
20	R_1	0.9432	0.8463	0.8456	0.473	0.5767	0.5765
	R_2	0.9664	0.8703	0.8696	0.297	0.4781	0.4785
	R_3	0.9588	0.8636	0.8629	0.3532	0.5258	0.5261
	R_4	0.9323	0.84	0.8393	0.5185	0.6081	0.6079
30	R_1	0.9088	0.8264	0.8257	0.5777	0.6585	0.6585
	R_2	0.9408	0.8559	0.8553	0.4387	0.5936	0.5942
	R_3	0.9095	0.8288	0.8281	0.5629	0.6675	0.6679
	R_4	0.8998	0.8198	0.8192	0.5935	0.6703	0.6704
32	R_1	0.9073	0.8281	0.8275	0.5708	0.6722	0.6725
	R_2	0.9255	0.8445	0.8438	0.5157	0.6498	0.6503
	R_3	0.9068	0.8277	0.8271	0.5768	0.6802	0.6805
	R_4	0.9049	0.824	0.8234	0.5859	0.668	0.6682
40	R_1	0.8821	0.809	0.8084	0.6168	0.6895	0.6898
	R_2	0.8959	0.8214	0.8208	0.6044	0.7038	0.7042
	R_3	0.891	0.8168	0.8162	0.6003	0.6929	0.6933
	R_4	0.9063	0.8316	0.831	0.5823	0.6953	0.6958

Table 8: Risk of the estimated values for the parameters θ & β under different estimation methods when their values are $\theta = 2$ and $\beta = 2$

m	RS	$risk(\hat{\theta}_{ML})$	$risk(\hat{\theta}_{BS})$	$risk(\hat{\theta}_{BL})$	$risk(\hat{\beta}_{ML})$	$risk(\hat{\beta}_{BS})$	$risk(\hat{\beta}_{BL})$
20	R_1	1.2831	0.0551	0.0003	0.7833	0.132	0.0007
	R_2	2.6207	0.0424	0.0002	0.4964	0.2356	0.0012
	R_3	2.0357	0.0468	0.0002	0.5653	0.1625	0.0008
	R_4	0.8551	0.0673	0.0003	0.6019	0.1057	0.0005
30	R_1	0.3226	0.0906	0.0005	0.2521	0.0759	0.0004
	R_2	0.9662	0.0438	0.0002	0.3484	0.0814	0.0004
	R_3	0.3582	0.0945	0.0005	0.2091	0.0647	0.0003
	R_4	0.2453	0.1122	0.0006	0.185	0.0816	0.0004
32	R_1	0.3146	0.0916	0.0005	0.1971	0.059	0.0003
	R_2	0.5852	0.0623	0.0003	0.3152	0.0532	0.0003
	R_3	0.318	0.0957	0.0005	0.2143	0.0567	0.0003
	R_4	0.2893	0.0988	0.0005	0.2212	0.0716	0.0004
40	R_1	0.1062	0.1382	0.0007	0.0751	0.0805	0.0004
	R_2	0.1979	0.1065	0.0005	0.1527	0.0474	0.0002
	R_3	0.1597	0.1197	0.0006	0.0999	0.0599	0.0003
	R_4	0.2713	0.0784	0.0004	0.2058	0.039	0.0002

Table 9: CIs for estimated values of θ under different estimation methods

m	RS	Boot-p	Boot-t	HPD
20	R_1	(2.8578, 11.1488)	(2.6288, 6.1536)	(1.3261, 2.4467)
	R_2	(3.7472, 18.0335)	(3.7067, 10.1156)	(1.4354, 2.6929)
	R_3	(3.4901, 13.8735)	(3.3244, 8.5)	(1.3958, 2.6221)
	R_4	(2.7133, 8.034)	(2.366, 5.0645)	(1.2945, 2.3969)
30	R_1	(2.0618, 5.1748)	(1.8705, 3.9422)	(1.2531, 2.2653)
	R_2	(2.7957, 8.0421)	(2.6447, 6.0912)	(1.3757, 2.5225)
	R_3	(2.2813, 4.6984)	(1.8374, 3.4712)	(1.2465, 2.3016)
	R_4	(2.042, 4.5805)	(1.7029, 3.3854)	(1.22, 2.2205)
32	R_1	(2.1664, 4.5572)	(1.824, 3.4918)	(1.2536, 2.2879)
	R_2	(2.4742, 5.755)	(2.2791, 4.7645)	(1.3244, 2.4148)
	R_3	(2.2539, 4.5638)	(1.858, 3.504)	(1.2478, 2.2867)
	R_4	(1.9942, 4.8306)	(1.8056, 3.7792)	(1.2455, 2.2427)
40	R_1	(1.7312, 3.5425)	(1.5848, 3.0472)	(1.1956, 2.1346)
	R_2	(1.9907, 3.9013)	(1.7448, 3.2471)	(1.235, 2.2288)
	R_3	(1.9355, 3.6744)	(1.6093, 2.973)	(1.2116, 2.1957)
	R_4	(2.0731, 4.3427)	(1.9664, 3.8268)	(1.2814, 2.3017)

Table 10: CIs for estimated values of β under different estimation methods

m	RS	Boot-p	Boot-t	HPD
20	R_1	(3.1851, 6.4733)	(2.077, 4.0564)	(1.0733, 2.3297)
	R_2	(2.3496, 4.8528)	(1.9902, 3.9582)	(1.0065, 2.1425)
	R_3	(2.218, 4.4424)	(2.379, 4.5623)	(1.0882, 2.2594)
	R_4	(2.6085, 5.1819)	(2.2945, 4.4122)	(1.126, 2.3731)
30	R_1	(2.8568, 4.9921)	(1.6458, 2.892)	(1.2644, 2.3197)
	R_2	(2.4732, 4.1915)	(2.1725, 3.5995)	(1.2992, 2.3392)
	R_3	(1.6349, 2.9493)	(2.5221, 4.5128)	(1.3172, 2.349)
	R_4	(2.3164, 3.9038)	(1.7931, 3.0022)	(1.2702, 2.2951)
32	R_1	(1.9933, 3.2423)	(2.3841, 3.8558)	(1.3405, 2.3472)
	R_2	(2.2417, 3.4711)	(2.6081, 3.9883)	(1.3807, 2.4114)
	R_3	(1.7294, 2.9539)	(2.5262, 4.2396)	(1.3539, 2.3694)
	R_4	(2.728, 4.793)	(1.5904, 2.7698)	(1.2914, 2.3134)
40	R_1	(2.3153, 3.6335)	(1.4744, 2.3389)	(1.3206, 2.2107)
	R_2	(1.9488, 2.9533)	(2.2802, 3.448)	(1.4162, 2.3473)
	R_3	(1.7508, 2.7971)	(2.0975, 3.3296)	(1.3706, 2.2749)
	R_4	(2.1957, 3.3477)	(2.3853, 3.5826)	(1.4553, 2.4036)

Table 11: Estimates of parameters (θ,β) for various choice of parameters under different estimation methods for $m=32$ failures

θ	β	RS	$\hat{\theta}_{ML}$	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	$\hat{\beta}_{ML}$	$\hat{\beta}_{BS}$	$\hat{\beta}_{BL}$
1.5	2	R_1	1.688	1.4188	1.4165	0.1152	0.0381	0.0002
	2	R_2	1.8131	1.4962	1.4936	0.2111	0.0397	0.0002
	2	R_3	1.6819	1.412	1.4097	0.1115	0.0382	0.0002
	2	R_4	1.6885	1.4149	1.4127	0.119	0.0376	0.0002
3	2	R_1	3.8175	2.2328	2.2261	1.2067	0.6187	0.0031
	2	R_2	4.2918	2.3469	2.3394	2.4905	0.4562	0.0023
	2	R_3	3.8472	2.2421	2.2354	1.3377	0.6097	0.003
	2	R_4	3.7751	2.1794	2.173	1.1533	0.6995	0.0034
2	0.5	R_1	2.3617	1.8129	1.809	0.3032	0.0889	0.0004
	0.5	R_2	2.562	1.9253	1.9209	0.5415	0.0703	0.0003
	0.5	R_3	2.3462	1.7956	1.7916	0.3092	0.1006	0.0005
	0.5	R_4	2.3561	1.8219	1.8181	0.2799	0.0775	0.0004
2	1.5	R_1	2.3704	1.7738	1.7701	0.3032	0.0868	0.0004
	1.5	R_2	2.5964	1.8849	1.8807	0.5864	0.0609	0.0003
	1.5	R_3	2.3752	1.7754	1.7715	0.3486	0.1012	0.0005
	1.5	R_4	2.3506	1.7606	1.7571	0.3006	0.0946	0.0005

Table 12: Risk of the estimated values for the parameters (θ, β) for various choice of parameters under different estimation methods for $m = 32$ failures

θ	β	RS	$risk(\hat{\theta}_{ML})$	$risk(\hat{\theta}_{BS})$	$risk(\hat{\theta}_{BL})$	$risk(\hat{\beta}_{ML})$	$risk(\hat{\beta}_{BS})$	$risk(\hat{\beta}_{BL})$
1.5	2	R_1	0.1152	0.0381	0.0002	0.1984	0.0463	0.0002
	2	R_2	0.2111	0.0397	0.0002	0.2992	0.0477	0.0002
	2	R_3	0.1115	0.0382	0.0002	0.1912	0.0507	0.0003
	2	R_4	0.119	0.0376	0.0002	0.2282	0.051	0.0003
3	2	R_1	1.2067	0.6187	0.0031	0.2047	0.129	0.0006
	2	R_2	2.4905	0.4562	0.0023	0.3155	0.1182	0.0006
	2	R_3	1.3377	0.6097	0.003	0.1924	0.1377	0.0007
	2	R_4	1.1533	0.6995	0.0034	0.2064	0.1716	0.0009
2	0.5	R_1	0.3032	0.0889	0.0004	0.0119	0.0045	2.00E-05
	0.5	R_2	0.5415	0.0703	0.0003	0.0197	0.0067	3.00E-05
	0.5	R_3	0.3092	0.1006	0.0005	0.0121	0.0048	2.00E-05
	0.5	R_4	0.2799	0.0775	0.0004	0.0133	0.0041	2.00E-05
2	1.5	R_1	0.3032	0.0868	0.0004	0.1142	0.0262	0.0001
	1.5	R_2	0.5864	0.0609	0.0003	0.1638	0.027	0.0001
	1.5	R_3	0.3486	0.1012	0.0005	0.1198	0.0272	0.0001
	1.5	R_4	0.3006	0.0946	0.0005	0.1189	0.0296	0.0001

Table 13: Estimated values of survival and hazard rate functions at specific time $t = 1$ for various choice of parameters and for $m = 32$ failures

θ	β	RS	$\hat{R}_{ML}$	$\hat{R}_{BS}$	$\hat{R}_{BL}$	$\hat{h}_{ML}$	$\hat{h}_{BS}$	$\hat{h}_{BL}$
1.5	2	R_1	0.8151	0.758	0.7574	0.9002	0.8769	0.8764
	2	R_2	0.8369	0.776	0.7754	0.8692	0.8645	0.8642
	2	R_3	0.814	0.7563	0.7558	0.8961	0.8778	0.8773
	2	R_4	0.8152	0.7571	0.7565	0.9094	0.8629	0.8623
3	2	R_1	0.978	0.8928	0.8921	0.2021	0.4485	0.4498
	2	R_2	0.9863	0.9043	0.9036	0.1465	0.4209	0.4223
	2	R_3	0.9787	0.8938	0.8931	0.1954	0.4425	0.4437
	2	R_4	0.9771	0.8869	0.8862	0.2081	0.4475	0.4487
2	0.5	R_1	0.9057	0.8368	0.8362	0.1435	0.1877	0.188
	0.5	R_2	0.9228	0.8542	0.8535	0.1319	0.1812	0.1816
	0.5	R_3	0.9043	0.834	0.8333	0.1448	0.1909	0.1912
	0.5	R_4	0.9052	0.8383	0.8377	0.1453	0.1834	0.1838
2	1.5	R_1	0.9066	0.8303	0.8297	0.4314	0.5262	0.5267
	1.5	R_2	0.9255	0.8482	0.8475	0.3841	0.5006	0.5013
	1.5	R_3	0.907	0.8306	0.8299	0.4314	0.5286	0.5291
	1.5	R_4	0.9047	0.8281	0.8275	0.4387	0.5169	0.5172

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