



Il Colle di  
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# Statistical Mechanics, Integrability and Combinatorics

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**Abstract.** In Spring 2015, the Galileo Galilei Institute for Theoretical Physics hosted an eight-week Workshop on “Statistical Mechanics, Integrability and Combinatorics”. The Workshop addressed a series of questions in the realm of exactly solvable models of statistical mechanics, featuring numerous ties and overlaps with various problems in modern combinatorics, probability theory, and representation theory. Much recent progress in these areas exploits the underlying notion of quantum integrability. We report here on the scientific motivations and background for this activity and on its main outputs.

**Keywords.** Statistical mechanics, lattice models, quantum integrability, exact results, combinatorics, tilings, dimers, growth processes, limit shape phenomena, random matrices, random surfaces, determinantal processes, discrete holomorphicity.

## Scientific background

The last decade has witnessed increasing interactions and overlaps between the study of exactly solvable two-dimensional lattice models of statistical mechanics and various problems in modern combinatorics and probability theory. Newly-discovered links between ideas and tools in these different areas of mathematics and theoretical physics have led to numerous interesting results and a series of spectacular developments.

As examples of topics where such constructive interplay took place, we can mention: dimer models, random surfaces and limit shape phenomena; shape fluctuations and random matrices; random tilings, random partitions and stochastic growth processes; random tilings and representation theory; self-avoiding walks and Schramm-Loewner evolution; the idea of discrete holomorphicity and its relation with the notion of quantum integrability; a rigorous characterization of the Kardar-Parisi-Zhang universality class; the recently-developed notion of integra-

ble probability; classical problems in combinatorics, such as the enumeration of Alternating Sign Matrices and plane partitions and their relation with physical models, via the Razumov-Stroganov correspondence; lattice supersymmetry and, in particular, supersymmetric quantum spin chains.

Essential ingredients of such recent progress in the area are, on the one hand, the interplay between discrete and continuous formalism and, on the other hand, the notion of quantum integrability, permitting the possibility of arriving at exact results. Recent developments in the above-mentioned topics, are not only highly relevant in themselves, but bear profound implications across many different fields: from traditional models of statistical mechanics, disordered systems, classical and quantum non-equilibrium phenomena to algebraic geometry, string theory, quantum gravity and random matrices. Let us discuss briefly some of the above mentioned topics.

### Random tilings

The tiling of a surface is its covering with tiles of a defined shape, with no overlaps and no gaps. Given a finite two-dimensional region and a set of tiles, an interesting and relevant question is the statistical mechanics of the model, that is, the study of the behaviour of various observables, when averaged over all possible tilings (possibly with some non-uniform measure). In particular, tilings in two dimensions provide an interesting class of discrete geometric models that exhibit critical behaviour (that is, in the thermodynamic limit their correlation functions are observed to decay algebraically with distance) and are thus expected to be described by conformal field theories and SLEs.

Random tilings with lozenges of uniform measure, being free-fermionic and describable in terms of non-intersecting lattice paths, were analyzed in great detail (together with their limit shapes, emerging in the scaling limit), using techniques of solvable lattice models, combinatorics and random matrix theory. On the other hand, not much is known concerning more complex random tiling models that go beyond the class of free-fermion models, but are still Bethe Ansatz integrable, such as, for example: i) a class of tilings with interacting dominoes (i.e., with non-uniform measure), directly related to the six-vertex model; ii) a class of rectangle-triangle tiling models discovered in the late nineties, and with a recently unveiled connection with representation theory via Littlewood-Richardson coefficients. Further understanding of these models, including the determination of their limit shapes and, more generally, the full characterization of their behaviour in the scaling limit, would be highly desirable.

### Integrability and combinatorics

Several long-standing combinatorial problems have been solved in recent years, exploiting their connections to various two-dimensional exactly solvable mo-

dels. One of the key methods that has emerged in this context in the last decade is the use of quantum integrability and in particular of the quantum Knizhnik-Zamolodchikov equation, which is itself deeply related to representation theory. Advances in the last few years have provided proofs of some old conjectures as well as completely new results.

Novel directions that should be pursued include: further applications to enumerative combinatorics such as proofs of various open conjectures relating to Alternating Sign Matrices, plane partitions and their symmetry classes; a full understanding of the quantum integrability of the Fully Packed Loop model, a specific loop model at the heart of the Razumov-Stroganov correspondence; a complete description of the Brauer loop scheme, including its defining equations, and of the underlying poset; the extension of the relation between the loop model and the six-vertex model the case of the eight-vertex model (corresponding to elliptic solutions of the Yang-Baxter equation).

### Integrable probability

Some probabilistic models can be analyzed using essentially algebraic methods. A well-known example of such systems is the problem of finding the asymptotic distribution of the sum of a large number of independent, identically-distributed, random variables, when the pre-limit distribution is explicitly given. Under certain hypothesis, this leads to the celebrated central limit theorem.

In recent years, a similar line of thought is being extended to a certain class of stochastic systems, for which a “law of large numbers” still applies, with the Gaussian law replaced by the Tracy-Widom distributions. These were first observed in the study of spectral properties of large random matrices and were subsequently found in many different systems, such as: directed polymers in (1+1) dimensions; fluctuating non-equilibrium interfaces in the Kardar-Parisi-Zhang universality class; sequence matching problems in biology and in combinatorics; mesoscopic systems of condensed matters, and many others.

In the framework of physics, extending the whole probabilistic framework associated with the Gaussian law to the case of the Tracy-Widom distribution actually corresponds to extending currently available knowledge concerning free (or linear) systems to some classes of integrable or exactly solvable models. In other words, physics knowledge on quantum integrability is becoming precious for probabilists in their quest for more general probabilistic models. Viceversa, a deeper understanding of representation theoretic objects definitely supports further advances in quantum integrable and exactly solvable models. This constructive interplay has generated in recent years the new field of “integrable probability”.

## Discrete holomorphicity

Many important lattice models of statistical mechanics are exactly solvable or integrable. At the critical point of a continuous phase transition these models are expected to have conformal invariant scaling limits. The study of these systems combines conformal field theory to describe the scaling limit with the exact solution of the lattice model, based on the integrable structure obtained from solutions to the Yang-Baxter equations. Promising new mathematical ideas include conformal invariant random geometry, especially the random curves known as Schramm-Loewner evolutions. A key advance is the use of discrete harmonic or discrete holomorphic observables of the lattice models for proving the existence and conformal invariance of scaling limits.

In conclusion, the last decade has seen increasing interactions and overlaps between the study of exactly solvable two-dimensional lattice models of statistical mechanics and various problems in modern combinatorics and probability theory, with numerous newly-discovered links between ideas and tools in these different areas of mathematics and theoretical physics. This has in turn boosted a series of spectacular developments. As mentioned above, there are nevertheless still many open issues, and encouraging cross-fertilization promises to be fruitful in generating further progress in these areas.

## The Workshop

The Workshop followed similar programmes on the subject held over the last decade at the Newton Institute (Cambridge), the Schrödinger Institut (Wien), the CRM (Montreal), the Institut Poincaré (Paris), the MSRI (Berkeley), the Kavli Institute (Santa Barbara) and the Simons Center (Stony Brook).

The aims of the Workshop were to bring together experts in conformal field theory, integrable systems, analysis, probability theory, combinatorics and representation theory with the purpose of:

- i) representing the state of the art in a number of currently active areas of research in statistical mechanics and related to integrability, random geometry and combinatorics, as described above;
- ii) providing an opportunity for further cross-fertilization between these different areas, thus stimulating new ideas and further advances;
- iii) giving young researchers the opportunity to learn about the exciting new developments in these areas, and to be encouraged to contribute to them.

The main topics of the Workshop included:

- Random tilings and limit shape phenomena;
- Random matrices, determinantal processes and KPZ universality class;
- Discrete holomorphicity and integrability;

- Lattice models and combinatorics;
- Quantum integrability and correlation functions.

The eight-week Workshop was attended by around 130 participants, including Fields medallist Andrei Okounkov (Columbia) and many other leading scientists in theoretical and mathematical physics, and about 40 early stage researchers (i.e. graduate students or young post-doctoral fellows).

The Workshop scheduled about one research talk per day and one discussion session on a specific topic each week, leaving most of the time free for informal discussions and collaboration.

The Workshop also hosted a Focus Week on “Lattice Models: Exact Methods and Combinatorics” on May 18-22, and a Conference on “Random Interfaces and Integrable Probability” on June 22-26. Each of these events, scheduling about twenty one-hour research talks as well as a Poster Session to offer early-stage scientists an opportunity to present their work, gathered over 70 participants. Experts in the field can appreciate the number of distinguished scientists appearing in the list of speakers.

As for the scientific output of the Workshop, it is undoubtedly of considerable significance. The relaxed and informal atmosphere at the GGI during the Workshop encouraged discussions and collaboration.

Significant progress was made on several of the Workshop topics. Some results have already been published and many others will be in the coming months.

In conclusion, the Workshop was timely and successful; it strengthened interactions between the different communities and boosted further progress in the field, leading to the exploration of open issues and new research directions. The above communities are planning to gather again in 2016 for similar Workshops, to be held at the Simons Center (Stony Brook), and at the ITP (Natal), thus confirming its vitality and scientific impact.

## Acknowledgements

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