Il Colle di Galileo

Resurgence and Modularity in QFT and String Theory

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Abstract. Modularity is a particularly powerful symmetry with many applications in theoretical physics and mathematics, in particular quantum field theory and string theory. Resurgence is a general framework for advanced asymptotics which effectively unifies perturbative and non-perturbative physics via an intricate network of algebraic relations known as Ecalle's bridge equations. Modularity and resurgence overlap strongly in the exploration of non-perturbative completions of perturbative data, leading to profound new insights into the connection between weak coupling and strong coupling, and in the study of dualities, black holes and quantum gravity. The goal of this GGI Workshop was to bring together experts from these two frontier areas in theoretical and mathematical physics to foster further interaction between different scientific communities and to develop novel strategies to address truly significant problems.

Keywords: modular invariance, *S*-duality, exact asymptotics, resurgence analysis, non-perturbative effects.

The programme has been organised by
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- Michael Green (DAMTP, Cambridge);
- Luca Griguolo (University of Parma);
- Sarah Harrison (Northeastern University);
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- Axel Kleinschmidt (Albert Einstein Institute, Potsdam);
- Sameer Murthy (King's College London);
- Anne Taormina (Durham University);
- **Local organiser:** Li Gan (GGI).

Scientific context and motivations

Perturbation theory is a widely used mathematical approach in theoretical physics which offers approximate solutions to complex problems by introducing small corrections to a known solvable system. When an exact solution to a physical problem is unattainable, we can try to reduce it to an easier, oversimplified

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scenario, such as modelling a cow as a spherical-cow. The problem we wish to solve can then be solved by adding a tower of smaller and smaller perturbations to the easier, solvable starting point. This method allows us to systematically expand quantities of interest as perturbative series in terms of a small parameter governing the size of the corrections. Perturbation theory plays a crucial role in quantum mechanics, quantum field theory and statistical physics, where it helps describe interactions, compute energy levels and predict physical phenomena in complex systems. The task of understanding and making sense, i.e. resumming, such infinite towers of perturbations goes by the name of asymptotic analysis.

Asymptotics is one of the most powerful mathematical tools in theoretical physics, and resurgent asymptotics ("resurgence") is a modern formalism that goes far beyond traditional Poincare asymptotics. Mathematically, resurgence defines a class of "analysable" functions (Ecalle), more general than meromorphic, and for which Borel summation, monodromy and analytic continuation properties are well-defined, being encoded in a "trans-series" that unifies perturbative series expansions with exponential (and logarithmic) non-perturbative terms. While an asymptotic series contains only partial information about a function, its trans-series encodes the complete information via Ecalle's alien calculus and bridge equations. Besides the conceptual importance of reconstructing non-perturbative data out of perturbative expansions, the role of Ecalle's theory is paramount for the understanding of global properties of physical observables, for example their monodromy structure as functions of a complexified coupling constant or their behaviour at strong coupling. This is precisely the meeting point of resurgence and modularity in string theory and quantum field theory.

Resurgence is well established in applications to differential, difference and integral equations, both linear and nonlinear, resulting in profound consequences such as non-linear Stokes phenomena and wall-crossing, as well as practical exponential asymptotics. Resurgence is general and constructive, and since perturbative expansions in physics are generically asymptotic, the ideas and methods are directly applicable to gauge theory and string theory. Resurgence has been shown to be particularly powerful in the analysis of matrix models, topological string theory, Chern-Simons theories and localisable QFTs, theories in which modularity is also a key player.

A dramatic consequence of resurgence is that, in saddle point expansions, the all-orders fluctuations around different saddle points are generally quantitatively related to one another. This is a well-established feature of ordinary exponential integrals, but is still mysterious for infinite dimensional exponential integrals such as path integrals. Chern-Simons theories are gauge theories that act as delicate probes of the geometry and topology of the 3-manifolds on which they are defined, and it has been shown that mock modularity plays a particularly important role in understanding resurgent behaviour for Chern-Simons path integrals defined on certain classes of 3-manifolds. Furthermore, this connects directly

with applications in 2d conformal field theories, in supersymmetric gauge theories in 3 dimensions, in knot theory and in number theory.

Symmetry has long been a vital cornerstone for constraining, computing and understanding physical observables at all scales. Modular symmetry in string theory and quantum field theory has led to deep new connections between theoretical physics and mathematics. In its simplest form, modularity allows us to connect between the two opposite ends of weak- and strong-coupling for the same physical theory. Recent work has made it clear that non-perturbative effects constructed via resurgent analysis contain detailed information about global properties of physical quantities under the action of the modular group. This interplay between non-perturbative effects, weak/strong coupling dualities and different incarnations of modularity spans a wide range of physical theories. For example, the partition functions of supersymmetric three-dimensional field theories display a beautiful set of inter-relations between physics, topology and number theory. These physical quantities can be expanded at weak-coupling, i.e. large Chern-Simons level, where we know that very finely tuned non-perturbative corrections must be present in order to produce Mock-modular invariant partition functions.

In four dimensions we have a plethora of theories, starting with the ever-present $\mathcal{N} = 4$ SYM, where modularity dramatically constrains all physical observables. Modularity, when combined with a resurgence approach to supersymmetric localisa tion provides extremely fertile ground for solving an extensive class of four-dimensional supersymmetric field theories. From a string theory perspective, we have a multitude of examples where modularity intertwines with non-perturbative corrections and hence with resurgence analysis. The low-energy expansion of genus-one string theory amplitudes can be constructed in terms of an infinite class of modular forms usually named Modular Graph Functions. These building blocks are non-holomorphic, modular covariant functions of the modular parameter, τ, of the world-sheet torus. Amongst the many fascinating properties of such objects is that their asymptotic "weak-coupling" expansions near the cusp, $\tau \to i\infty$, include multiple zeta values and display an intricate network of algebraic and differential relations. Another instance of modularity in string theory arises when considering the non-perturbative U-duality group of ten-dimensional type IIB string theory. In this case, the modular group acts on the complexified string coupling, relating perturbative and non-perturbative effects in g_{string}. This symmetry is extremely constraining, allowing us to study higher curvature corrections to the four-graviton sector in terms of certain automorphic forms.

Finally, in quantum gravity, the entropy of a black hole is closely connected with the quantum degeneracies associated with its horizon. These counting functions are in principle computable, using a holographic perspective, in terms of string theory data and are therefore expected to be modular functions. However, wall-crossing phenomena which can be understood from a resurgent point of view in terms of the global properties of these functions, dramatically change the nature

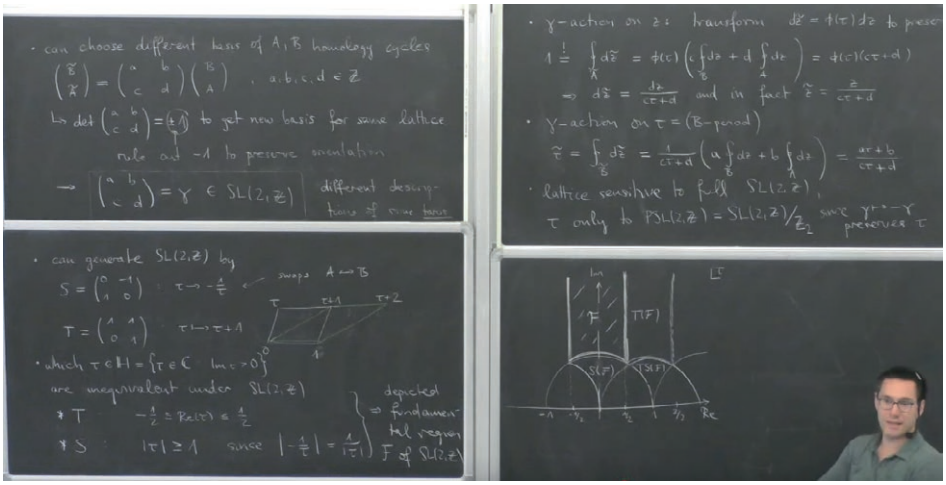

Figure 1. Prof. Oliver Schlotterer (Uppsala University) presenting some excellent lectures on modular invariance in string theory.

of these counting functions to Mock-modular forms, objects of fundamental importance for the moonshine programme.

Scientific activies

The scientific activity was structured around an initial "Training week", four regular "Workshop weeks" and a final "Conference week".

Training Week

The Training week was aimed at PhD students and early career researchers unfamiliar with this specific field of research but eager to grasp its fundamental concepts. The goal was to provide a basic introduction to the various aspects of modular invariance and resurgence analysis in the broader quantum field theory and string theory context. Ines Aniceto (Southampton) provided an introductory course to Ecalle's resurgence theory, Giulio Bonelli (SISSA) discussed exact methods in supersymmetric gauge theories, Oliver Schlotterer (Uppsala) presented an excellent overview of the power of modular forms for string theory which synergised fantastically with the course held by Roberto Volpato (Padua) on Mock Modularity and Moonshine from a string theory perspective. Figure 1 shows Prof. Oliver Schlotterer, one of the Simons fellows involved in the programme, lecturing at the blackboard. The students appreciated both the quality of the lectures provided and the time left for discussions with the lecturers, who were all absolutely brilliant and keen to engage with all the participants.

Figure 2. Participants of the conference week gathered in the sunny Cloister of the Galileo Galilei Institute.

Regular Workshop Activities

The planned workshop activities consisted of one/two invited talks every day, without any time limit. The quality of the talks was outstanding and it prompted many interesting discussions as well as future collaborations, intertwining the different areas of expertise of the audience. The various topics discussed included:

- Modular graph forms and the low-energy expansion of string theory.
- Resurgence, duality and modularity in AdS/CFT.
- Automorphic forms and non-perturbative scattering amplitudes.
- Resurgence and (Mock)-modularity in 3d Chern-Simons theories.
- Exact results in localizable superconformal quantum field theories.
- Resurgence and modularity in black holes and quantum information.

Conference Week

The conference week ran during the penultimate week of the programme and was attended in person by around 50 participants and featured a full-day schedule, comprising two talks in the morning and one talk in the afternoon, leaving plenty of time for discussion during the very active coffee breaks and lunch break. The range of talks covered all the most recent developments in the application of modular invariance and resurgence theory, with some key-note speakers presenting results in pure mathematics as well, further stimulating cross-fertilisation among the different areas of research. Figure 2 features a photo taken during the

conference week. All the recordings for the training week lectures and seminars are available on the GGI YouTube channel.

Final remarks

The workshop ran smoothly and without any issue, largely thanks to the fantastic support staff at GGI, who helped us massively during the preparation stage and throughout the duration of the programme. Praises regarding the outstanding lunches and the two "apericena" events which took place during the training week and the conference week were also received from many participants.

In short, we believe that the whole programme was a huge success and all the organisers would like to thank the GGI and INFN for funding this initiative, along with the directors and staff of the GGI for their invaluable support and dedication, and, of course, all the attendees for their constructive and collaborative participation.