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Reflections on Mathematics and Aesthetics

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> My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful.
>
> Hermann Weyl

> I am interested in mathematics only as a creative art.
>
> G. H. Hardy

1. *Mathematics as embodying intelligible beauty*

The association of mathematics with the Beautiful was first explicitly made by Plato (through the words of Socrates). In the *Philebus* Socrates says:

> I do not mean by beauty of form such beauty as that of animals or pictures... but ... understand me to mean straight lines and circles... for these I affirm to be not relatively beautiful, like other things, but eternally and absolutely beautiful.

For Plato the essence of mathematical beauty was its *absoluteness,* its resistance to change and fashion.

Some four centuries later, Plutarch asserts: «The purpose of geometry is to draw us away from the sensible and the perishable to the intelligible and eternal».

The word «intelligible» has two meanings. First, of course, the usual meaning of «comprehensible» or «capable of being understood». But the word also has an older meaning, namely, «capable of being apprehended only by the intellect, not by the senses»; in this guise it serves as an antonym to «sensible». It is, I believe, precisely with this signification that Plutarch uses the word «intelligible» in this quotation.

Plutarch doesn't mention «beauty» here, but he does link «sensible» with «perishable2 and «intelligible» with «eternal». Now most of those who discern beauty in mathematics (largely, but not exclusively mathematicians) would, I believe, hold the view that

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he kind of beauty involved is of the intelligible variety rather than the sensible. Thus Bertrand Russell:

Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spiritual delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry.

And G. H. Hardy:

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.

Many, I guess, would agree with Plato and Plutarch that the beauty of mathematics is grounded in its eternality, its permanence, that «cold, austere beauty». They would hold that mathematical truth, once established, becomes imperishable, just as Michaelangelo's «David», once carved from the marble, is held by artists to achieve an imperishability transcending its origins as an extract from a piece of stone, even should the statue itself be destroyed or crumble into dust. Hardy expresses this conviction memorably:

Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. «Immortality» may be a silly word, but probably a mathematician has the best chance of whatever it may mean.

The great 20th century French mathematician Claude Chevalley regarded the imperishability – and the aesthetic value – of mathematics as stemming from its employment of rigorous argument. As reported by his daughter Catherine:

For [my father] it was important to see questions as a whole, to see the necessity of a proof, its global implications. As to rigour, all the members of Bourbaki cared about it: the Bourbaki movement was started essentially because rigour was lacking among French mathematicians, by comparison with the Germans, that is the Hilbertians. Rigour consisted in getting rid of an accretion of superfluous details. Conversely, lack of rigour gave my father an impression of a proof where one was walking in mud, where one had to pick up some sort of filth in order to get ahead. Once that filth was washed away, one could get at the mathematical object, a sort of crystallized body whose essence is its structure. When that structure had been produced, he would say it was an object which interested him, something to look at, to admire, perhaps to turn around, but certainly not to transform. For him, rigour in mathematics consisted in making a new object which could thereafter remain unchanged.

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The way my father worked, it seems that this was what counted most, this production of an object which then became inert – dead, really. It was no longer to be altered or transformed. Not that there was any negative connotation to this. But I must add that my father was probably the only member of Bourbaki who thought of mathematics as a way to put objects to death for aesthetic reasons.

Thus in Chevalley's eyes, mathematical rigour had an aesthetic purpose, namely to induce a kind of *rigor mortis* in the objects of mathematics, so that their beauty could be put permanently on display, like butterflies mounted in a case[1].

But, as with artists, not all mathematicians subscribe to the doctrine that the purpose of their activity is to produce works of enduring unchangeability. One of the most important dissenters among mathematicians was Brouwer, the founder of the mathematical school of intuitionism, who once remarked that he had changed his view of mathematics as a collection of «truths fascinating by their immovability, but horrifying by their lifelessness, like stones from barren mountains of disconsolate infinity» to a concern with what is «built out of our own thinking».

In his *Consciousness, Philosophy and Mathematics* of 1948, Brouwer offers the following observations on beauty in general and in mathematics in particular:

In causal thinking and acting beauty will hardly be found. Things as such are not beautiful, nor is their ·domination by shrewdness. Therefore satisfaction at efficacy of causal acts or systems of causal acts or at discoveries of new causal sequences is no sensation of beauty.

But in the first phase of the exodus of consciousness from its deepest home there is beauty in the joyful miracle of the self-revelation of consciousness, as apparent in egoic elements of the object found in forms and forces of nature, in particular in human figures and human destinies, human splendour and human misery.

And in the second and third phase there is beauty in remembrance of the miracle of bygone naivety, remembrance evoked either by reverie through a haze of wistfulness and nostalgia, or by (self-created or encountered) works of art, or by certain kinds of science. Such science evoking beauty reveals or playfully mathematizes naively perceptible forms and laws of nature, after having approached them with attentive reverence, and with a minimum of tools. And such science evoking beauty, through its very reverence, rejects expansion of human domination over nature. Furthermore in the second and the third phase there is constructional beauty, which sometimes appears when the activity of constructing things is exerted

[1] For Chevalley mathematical rigour seems also have a moral significance. This is attested by the final sentence of the Preface to his work *Fundamental Concepts of Algebra,* which I studied in my youth, and has always remained with me: *this is an exercise in rectitude of thought, of which it would be futile to disguise the austerity.*

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playfully, and, thus getting a higher degree of freedom of unfolding, creates things evoking sensations of power, balance, harmony, and acquiescence with the exterior world.

But the fullest constructional beauty is the introspective beauty of mathematics, where instead of elements of playful causal acting, the basic intuition of mathematics is left to free unfolding. This unfolding is not bound to the exterior world, and thereby to finiteness and responsibility; consequently its introspective harmonies can attain any degree of richness and clearness.

Brouwer shared with the Romantics the conviction that beauty and truth are inextricably connected with life.

2. *Mathematics and music*

Mathematics has often been compared with music. For example, in a letter to Christian Goldbach of 1712, Leibniz remarks: «Music is a hidden arithmetic exercise of the soul, which does not know that it is counting».
Leibniz – ever the intellectual – actually went further in claiming that not only music, but the sensible in general is reducible to the intelligible:

Even the pleasures of sense are reducible to intellectual pleasures, known confusedly. Music charms us, although its beauty consists only in the agreement of numbers and in the counting, which we do not perceive but which the soul nevertheless continues to carry out, of the beats or vibrations of sounding bodies which coincide at certain intervals. The pleasures which the eye finds in proportions are of the same nature, and those caused by other senses amount to something similar, although we may not be able to explain them so distinctly. (Leibniz [1714]: 641)

For a long time Western scholars classified music (or its theoretical basis at least) as a branch of mathematics. This originated with the Pythagoreans who are said to have coined the term «mathematics» from a root meaning «learning» or «knowledge». The remarkable advances in mathematics made by the Pythagoreans led them to the belief that mathematics, and more especially *number*, lies at the heart of all existence – the first «mathematical» philosophy. For the Pythagoreans the structure of mathematics took the form of a bifurcating scheme of oppositions:

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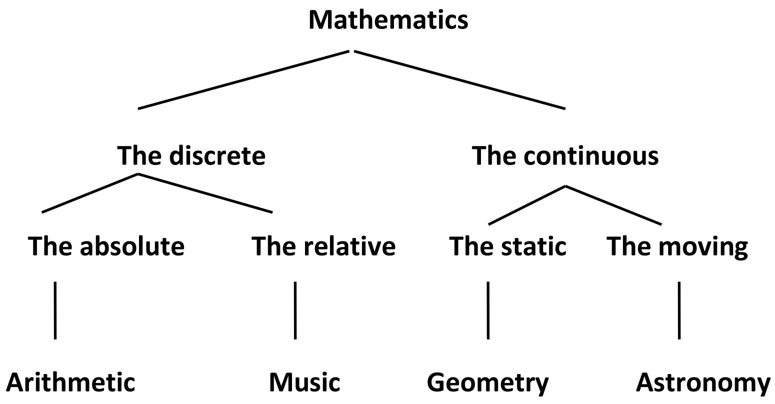

This scheme is the source of the *quadrivium*, which served as the basis for Western pedagogy until the end of the middle ages.

The Pythagoreans seem to have been concerned solely with the mathematical, i.e., intelligible, aspects of music. But of course as actually heard (or imagined) music has aesthetic qualities which are entirely sensible (*Heard melodies are sweet, but those unheard are sweeter*). Music pleases or moves us – it has an emotional content. Leibniz acknowledges this when he observes:

We do not always observe wherein the perfection of pleasing things consists, or what kind of perfection within ourselves they serve, yet our feelings perceive it, even though our understanding does not. We commonly say, «There is something, I know not what, that pleases me in the matter». This we call «sympathy». But those who seek the causes of things will usually find a ground for this and understand that there is something at the bottom of the matter which, though unnoticed, really appeals to us.

Music is a beautiful example of this. Everything that emits a sound contains a vibration or a transverse motion such as we see in strings; thus everything that emits sounds gives off invisible impulses. When these are not confused, but proceed together in order but with a certain variation, they are pleasing; in the same way, we also notice certain changes from long to short syllables, and a coincidence of rhymes in poetry, which contain a silent music, as it were, and when correctly constructed are pleasant even without being sung. Drum beats, the beat and cadence of the dance, and other motions of this kind in measure and rule derive their pleasurableness from their order, for all order is an aid to the emotions. And a regular though invisible order is found also in the artfully created beats and motions of vibrating strings, pipes, bells, and indeed, even of the air itself, which these bring into uniform motion. Through our hearing, this creates a sympathetic echo in us, to which our animal spirits respond. This is why music is so well adapted to move our minds, even though this main purpose is not usually sufficiently noticed or sought after. (Leibniz [1690-1698]: 425ff)

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One might go so far as to characterize music as «audible mathematics inducing pure emotion».

The sensible and intelligible qualities of music not only coexist, but can be experienced simultaneously. The pleasure of listening to a Bach fugue, for example, has both sensible and intelligible elements. In addition to responding to the purely sensible beauty of the music, an acquaintance with fugal form enables one to derive intellectual pleasure from hearing the structured entries of the subject, the countersubject, the stretto, etc. In a sense, the simultaneous presentation of the sensible and intelligible aspects of music provides the basis of the Pythagorean account of music as a branch of mathematics. For instance, consider their discovery that the euphony of the perfect fifth (*diapente*) is associated with the simple mathematical ratio 3: 2. Both are beautiful – the former to the senses, the latter – if in an elementary way – to the intellect. The sensory quality is directly revealed to the senses, not the intellect, while the mathematical quality is directly grasped by the intellect, not the senses – yet after the Pythagorean revelation both became graspable by the intellect simultaneously. Of course, the two may evoke one another as a mere linkage: it is perfectly possible for one to hear in one's inner ear a perfect fifth chord when contemplating the ratio 3:2. In fact, this phenomenon arises quite commonly with opus numbers: in my case for instance, the number «59» immediately evokes the beginning of Beethoven's first Razumovsky quartet, and «511» the first few bars of Mozart's A minor rondo.

The *equal temperament* scale (famously used by Bach in composing his *48 Preludes and Fugues*) provides a major link between mathematics and music. Here, the intervals between the notes of the 12 pitch chromatic scale are made uniform by specifying that the pitch ratio of each note to its predecessor is exactly $2^{1/12}$. The result is a scale completely symmetric in ascent and descent, so that one can start with any pattern of notes whatsoever and transpose or invert it at will without harmonic distortion.

In the last century, the great Austrian composer Arnold Schoenberg, in creating his dodecaphonic or serial method of composition, took full advantage of the musical possibilities offered by mathematical operations on note patterns. In Schoenberg's approach, the notes of the chromatic scale are initially arranged in a fixed order called a *series*. In the development of the composition the series may be manipulated in a variety of ways – for example, transposed, or inverted, or reversed. The series provides thus the germ from which the whole work grows.

Schoenberg's journey to serialism had begun some time earlier with his abandonment of traditional tonal composition and his revolutionary declaration of «the emancipation of the dissonance». But soon his revolutionary fervour gave way to the uncomfortable realization that in abandoning tonality he had opened the door to musical anarchy. In Schoenberg's eyes the anarchy unleashed by atonalism had nothing directly to do with the loss of the euphony of diatonic composition it entails – the loss, one might say, of sensible beauty. After all, there was nothing preventing an atonal composer from observing occasional adherence to tonal compositional practice – indeed the majority of atonal composers did exactly that. (As Schoenberg is supposed to have said, «there is still plenty of good music to be written in C major»). What seems to have concerned Schoenberg was the loss in the transition to atonality of the organizational principles governing tonal composition: major-minor triads, sonata form, and the like. In other words, he lamented the loss, not of sensible beauty but of *intelligible* beauty. Accordingly he sought new organizing principles for musical composition, once again, as with the Pythagoreans, based on mathematics. But this time, the Western compositional system, itself the result of piecemeal historical additions to the original Pythagorean insight that euphony as perceived by the human ear is built on simple arithmetical relations was to be replaced by an organized structure built from a single universal musico-mathematical entity – the equal temperament scale.

Another revival of the Pythagorean link between mathematics and music has appeared with the rise of so-called *musical set theory.* Here musical compositions are analyzed by treating them as sets, sequences and permutations of pitches or pitch-classes (equivalence classes under octaves) in equal temperament tuning subject to musical operations such as transposition, inversion, and complementation. Musical set theory arose naturally as a mathematical analysis of the serial method of composition. More recently, *mathematical music theory* has been developed in an attempt to extend the original Pythagorean analysis of euphony. Here category theory and topos theory have been pressed into service to explicate the use of the diatonic scale and the consonance/dissonance opposition.

3. *Mathematics and art: perspective*

There are numerous links between mathematics and the visual arts. The most important of these arose in the emergence of projective geometry from the study of perspective by visual artists: a rare example of a mathematical discipline whose origins lie entirely in art. Projective geometry issued from the development in the 15th and 16th centuries of a radically new approach to *perspective drwaing*. The central problem of perspective drawing that is, the portrayal three dimensions on a two dimensional surface, had been

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studied by artists since the Stone Age. For example, a fifteen thousand year old etching of a herd of reindeer on a bone fragment discovered by archaeologists creates the impression of distance by displaying the legs and antlers as if seen beyond the fully sketched animals of the foreground. The main perspective problem encountered by Egyptian artists was the portrayal of a single important object with the necessary dimension of depth: this was achieved in an ingenious manner by drawing a combination of horizontal and side view. Thus, for instance, in drawing a Pharaoh carrying a circular tray of sacrificial offerings, the top view of the tray is shown in half display by means of a semicircle, and on this half-tray is presented the sacrificial food as it would appear from above. This stylized method of expressing a third dimension persisted in Egyptian drawing for millenia. An arresting method of achieving this effect was created by American Indian artists who, in their drawings of persons or animals, present views of both front and left – and right hand sides. The figures are drawn as if split downs the back and flattened like a hide, with the result that each side of the head and body becomes a profile facing the other. Landscapes drawn by Chinese artists create the impression of space and distance by skillful arrangement of land, water and foliage. In drawing buildings, however, it was necessary to display the parallel horizontal lines of the construction, and for this the technique of *isometric drawing* was used. This is a simulation of perspective drawing in which parallel lines are drawn parallel, instead of converging as in true perspective.

In Europe, it was not until the first half of the fifteenth century that Italian painters, through the introduction of the *horizon line* and *vanishing point,* transformed perspective drawing into an exact science. The fundamental principles were worked out by the architect Filippo Brunelleschi and the artist and mathematician Piero della Francesca. The formulation of the laws of perspective revolutionized painting in Renaissance Italy, and the technique of perspective became an essential constituent in the works of later masters. In his great wall-painting *The Last Supper*[2], for example, Leonardo da Vinci employs perspective in a subtle way to draw the viewer's eye to the composition's centre. Albrecht Dürer's woodcut *Man Drawing a Lute* actually depicts two figures engaged in applying the theory of perspective. One figure examines how light (represented by a string attached to a wall on one end and marked on the edge of a lute on the other end) moves through a wooden frame attached vertically to a long table. The wooden frame is used to represent the image seen by a viewer looking directly at the lute. This study en-

[2] Happily, this masterpiece, in sadly deteriorated state for as long as anyone can recall, has been restored.

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ables the figures to produce a relatively accurate drawing of the object, as shown by the image held in the hand of the other figure.

The origins of *projective geometry* lie in the study of perspective. A painter's picture, or a photograph, can be regarded as a *projection* of the depicted scene onto the canvas or photographic film, with the painter's eye or the focal point of the camera's lens acting as the centre of projection. For example, suppose we take a photograph of a straight railway track, with equally spaced ties, going directly away from us. In the photograph the parallel lines of the rails appear to converge, meeting at a vanishing point or «point at infinity»; the equal spaces between the ties appear as unequal; and the right angles between the rails and the ties appear as acute. A circular pond in the landscape would appear as an ellipse. Nevertheless the geometric structure of the original landscape can still be discerned in the photograph. This is possible only if the original scene and its image have certain geometric properties in common, properties which, unlike lengths and angles, are preserved under the passage from the one to the other. These common geometric properties are called *projective* properties.

Projective geometry is the mathematical study of projective properties. Projective geometry is the mathematics of perspective, and in their use of perspective artists may be said to be presenting mathematics in a visual form.

Early in the 20th century, certain artists began to abandon what by this time had become the traditional technique of perspective. These artists – the Cubist painters led by Braque and Picasso – introduced a new technique which, instead of attempting to produce the impression of three dimensions, brought the bidimensionality of the canvas into the foreground. To achieve this they decomposed objects into elementary geometric forms such as planes, cubes and pyramids which were then reassembled and presented in a relieflike way on a flat or shallow space.

This technique dominates Picasso's early cubist paintings, in particular the famous *Les Demoiselles d'Avignon* (1907), with its radical distortion of figures and presentation as fractured planes. In 1908 Braque introduced geometric cubism, in which natural objects such as trees and mountains were presented as shaded cubes and pyramids. The art critic Louis Vauxcelles was piqued to describe these as «bizarreries cubiques» which apparently gave the movement its name.

A number of contemporary critics thought that in abandoning perspective the cubist painters were taking a step backwards. Certainly Vauxcelles thought so, referring to the members of the cubist movement as «ignorant geometers, reducing the human body, the site, to pallid cubes». From a mathematical standpoint, however, the cubists were

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actually making what has to be regarded as an advance. For their technique of decomposing objects into elementary geometric forms is closely analogous to the central idea of *combinatorial topology*, a branch of mathematics which was emerging at about the same time, led by the great French mathematician Henri Poincaré. Here the central idea is the investigation of the properties of topological spaces by subjecting them to combinatorial decomposition into simpler spaces such as *simplicial complexes,* constructed by «gluing together» points, line segments, triangles, and their higher dimensional counterparts. The planes, cubes and pyramids of the Cubist artists correspond to the simplicial complexes of the mathematicians.

4. *Symmetry*

An aspect of the arts in which mathematics plays a central role is in the use of *symmetry.* In everyday discourse the word is associated with a sense of harmonious proportion and balance: the idea of a whole composed of parts fitting together in an aesthetically pleasing way. As Pascal observes in the *Pensées,* «symmetry is what we see at a glance».

And Leibniz again:

> The pleasures of sense which most closely approach pleasures of the mind, and are the most pure and the most certain, are that of music and that of symmetry, the former [being pleasure] of the ears, the latter of the eyes; for it is easy to understand the principles [raisons] of harmony, this perfection which gives us pleasure. The sole thing to be feared in this respect is to use it too often. (Leibniz [1694-1698a]: 83)

And the English poet Anna Wickham:

God, Thou great symmetry,
Who put a biting lust in me
From whence my sorrows spring,
For all the frittered days
That I have spent in shapeless ways
Give me one perfect thing[3].

In mathematics, the term «symmetry» has a more precise definition involving the concept of *invariance* under a transformation or transformations. For example, a circle possesses full rotational symmetry in the plane since it is invariant under arbitrary rotations about its centre, and a rectangle is bilaterally symmetric since it is invariant under reflections about a central axis. A pattern such as *ABC ABC ABC*... exhibits translational or re-

[3] Quoted by Weyl (1952): Introduction.

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petitive symmetry since the foot *ABC* is reproduced by a left-to-right translation of the whole pattern.

Poetic metre[4] is built on the idea of repetitive symmetry. Here (in classical English poetry at any rate) each line of a poem exhibits a fixed pattern of «feet» consisting of stressed (or long) and unstressed (or short) syllables. Here is a table of the various meters:

Foot type	**Meter**	**Stress pattern**	**Syllable count**
Iamb	Iambic	Unstressed + Stressed	2
Trochee	Trochaic	Stressed + Unstressed	3
Spondee	Spondaic	Stressed + Stressed	2
Anapest or anapaest	Anapaestic	Unstressed + Unstressed + Stressed	3
Dactyl	Dactylic	Stressed + Unstressed + Unstressed	3
Amphibrach	Amphibrachic	Unstressed + Stressed + Unstressed	3
Pyrrhic	Pyrrhic	Unstressed + Unstressed	2

Shakespeare's famous sonnet, which begins:

Shall I compare thee to a summer's day?
Thou art more lovely and more temperate:
Rough winds do shake the darling buds of May,
And summer's lease hath all too short a date: (Shakespeare, Sonnet XVIII)

Is an example of *iambic pentameter,* in which the iambic foot *unstressed/stressed* is repeated 5 times in each line.

The beginning of Coleridge's poem *Kubla Khan*

In Xanadu did Kubla Khan
A stately pleasure-dome decree:
Where Alph, the sacred river, ran

[4] If, as Poincaré says, mathematics is the art of calling different things by the same name, then poetry is the art of calling the same thing by different names.

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Through caverns measureless to man
Down to a sunless sea.

Is an example – with the exception of the last line – of *iambic tetrameter* in which the iambic foot is repeated 4 times in each line.

In his brilliant poem *Metrical Feet,* Coleridge fashions examples of most of the poetic meters:

Trōchĕe trīps frŏm lōng tŏ shōrt;
From long to long in solemn sort
Slōw Spōndēe stālks; strōng foot! yet ill able
Ēvĕr tŏ cōme ŭp wĭth Dāctyl trĭsllăblĕ.
Ĭāmbĭcs mārch frŏm shōrt tŏ lōng;—
Wĭth ă leap ănd ă bound thĕ swĭft Ānăpæsts thrōng;
One syllable long, with one short at each side,
Ămphībrăchys hāstes wĭth ă stātely stride;—
Fīrst ănd lāst bēĭng lōng, mīddlĕ shōrt, Amphĭmācer
Strīkes hĭs thūndērīng hoofs līke ă proud hīgh-brĕd Rācer.

Repetitive symmetry is a central organizing principle in *music.* This form of symmetry plays a dominant role in the structure of rhythm, in which a rhythmic pattern such as 2 : 2, 3 : 4 (where the lower numeral indicates the note value that represents one beat, while the upper numeral indicates how many such beats occur in a bar) is repeated throughout a portion of a composition. Repetitive symmetry is a central feature of jazz and in the repeated AAB form that dominates the 12 bar blues. Many of the compositions of the minimalist composer Steve Reich – such as his *Music for 18 Musicians* – are vast repetitive symmetries built on the repetition of a musical phrase with tiny accumulating variations under which its form is gradually changed.

Fugues and *canons* provide further examples of repetitive symmetry in music.

Bilateral or *reflective* symmetry arises in many of the arts. Numerous examples of bilateral symmetry in visual art are offered in Hermann Weyl's great book *Symmetry.* In music and literature, compositions possessing bilateral symmetry are known as *palindromes.* Famous example of musical palindromes is the Minuetto *al Reverso* in Haydn's Symphony No. 47 and the *Adagio* in Berg's Chamber Concerto. In both of these compositions the musical line is exactly reversed at its midpoint. Palindromic forms also abound in the music of Anton Webern. The second movement of his serial *Piano Variations, Op. 27,* is an arresting instance of a *vertical palindrome,* in which *each* pitch of which is arranged symmetrically around a central pitch.

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Retrograde inversion, in which a musical phrase is reversed and inverted, is another musical device based on symmetry. Hindemith's *Ludus Tonalis* provides a nice example: here the preludium and postludium are in retrograde inversion. The *crab canon* is a particularly clever kind of retrograde inversion in which a musical line and its retrograde inverse are played simultaneously. The most famous example of crab canon is to be found in Bach's Musical *Offering.* The film-maker Jos Leys has brilliantly displayed the mathematical structure of Bach`s canon by displaying the score on a Möbius strip.

Arch form provides yet another instance of bilateral symmetry in music. Here the movements or sections of a work are arranged symmetrically around a central movement or section. Arch form had a particular appeal to Béla Bartok: it is to be found in his fourth and fifth string quartets, second piano concerto, and his concerto for orchestra.

The best-known form of palindrome in literature is the *letter-unit palindrome*, a phrase or sentence which reads the same when the order of the letters is reversed with general allowances for adjustments to punctuation and word dividers. Famous examples include *Able was I ere I saw Elba; A man, a plan, a canal - Panama!* The construction of palindromes appeals particularly to those with a mathematical turn of mind. The topologist Peter Hilton constructed a particularly ingenious example: *Doc note, I dissent. A fast never prevents fatness. I diet on cod.* A haunting example of a poem each line of which is palindromic is Leigh Mercer's *Four palindromes of the Apocalypse:*

An era, midst its dim arena
Elapses pale
No, in uneven union,
Liars, alas, rail.

Surprisingly, perhaps, much less common are *word-unit* palindromes, which read the same when the order of the words, rather than the letters, is reversed. A famous one is the motto of the Three Musketeers: *All for one and one for all.* J.A. Lindon has constructed a number of nice examples, such as: *Bomb disposal squad with failed technique failed with squad-disposal bomb.* An example of a word-unit palindrome with a mathematical flavour is: *You can circle the square, can't you, but you can't square the circle, can you?*

5. *The formal beauty of mathematics*

Inasmuch as beauty in mathematics does not appeal directly to the senses, it is often claimed to possess a *formal* character. This is witnessed by the common use by mathematicians of the term *elegant*, as in an elegant proof or an elegant solution.

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It is worth dwelling on the meaning of the term *formalism*. In art, for example, the term has been associated with the doctrine that each work of art contains within itself all the elements necessary for understanding and responding to it – an individual work of art is, so to speak, *sui generis.* On this account, to appreciate a work of art one need know nothing of the social or historical context in which the work was created[5]. In this sense mathematics certainly tends towards formalism, since it appears to have a *synchronic* as opposed to *diachronic* character – once the appropriate cognitive apparatus is in place, understanding mathematical propositions and proofs depends – like the assembling of jigsaws – only on grasping how the constituents «fit together».

The synchronic, formal character of mathematics is one natural source of its objectivity. For it leads us to liken the process of proving a mathematical proposition, or solving a mathematical problem to the fashioning of a key to open a lock. Once a lock has been constructed, it becomes an objective matter as to whether a particular key opens it, and also whether such a key can actually be fashioned. Analogously, once a mathematical proposition has been formulated, it becomes an objective matter as to whether a proposed proof is actually a proof. This can be a weighty issue: witness the effort it took for Andrew Wiles to produce a convincing proof of Fermat's Last Theorem.

Mathematicians see a formal beauty in *mathematical symbolism,* in which they discern both the intelligible beauty of the concepts expressed by the symbolism and the purely sensible beauty of the symbols themselves. This sensible beauty can appeal even to non-mathematicians, just as the purely visual beauty of Chinese calligraphy or Arabic script can appeal to those who are, like myself, ignorant of either language.

Some examples:

$$\begin{array}{c} 1 \\ 1\quad 1 \\ 1\quad 2\quad 1 \\ 1\quad 3\quad 3\quad 1 \\ 1\quad 4\quad 6\quad 4\quad 1 \\ 1\quad 5\quad 10\quad 10\quad 5\quad 1 \end{array}$$

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (bc - ad)^2 = (ad + bc)^2 + (ac - bd)^2,$$

[5] One notes that in both visual art and music the term *formalism* has often been employed with pejorative intent – as indicating that «content» has been sacrificed in favour of (empty) «form».

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$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

$$e^{i\pi} + 1 = 0$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \text{ (Equation of continuity)}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \text{ (Heat equation)}$$

$$d(uv) = udv + vdu \text{ (Leibniz's law)}$$

$$d\int f = f \text{ (Fundamental Theorem of the Calculus)}$$

$$\partial\partial = 0 \text{ (Nilpotence of the boundary operator)}$$

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \text{ (Metric equation in Riemannian geometry)}$$

6. *The timelessness of mathematics*

Mathematicians like to think that in practicing their art they are exploring (or creating) a perfect, timeless realm wholly free of the flaws and uncertainties besetting the empirical world. To quote Hardy once again:

I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our «creations», are simply the notes of our observations.

Mathematicians are moved – almost to the point of tears – to behold the timeless realm of mathematics, in which beauty is so closely allied to truth. Indeed, the first half of Keats' dictum, *Beauty is truth, truth beauty* is held by mathematicians to be realized in

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the mathematical realm. Not, however, the second half, since there are plenty of mathematical assertions which, while undeniably true, can hardly be described as beautiful. This fact points up one essential difference between the world of mathematics and the world of art or music – close as they are in many respects. The artist or musician has a high degree of control over the world he or she creates through the production of art or musical works. The artist can choose to create works of art that are surpassingly beautiful, or shockingly ugly (especially in contemporary art!) The mathematician has no such freedom (except in the production of expository works) for in the end he or she is constrained by the dictates of mathematical truth and proof. Here we recall Eddington's observation: *Proof is the idol before whom the pure mathematician tortures himself.* The artist can actually *set out* to produce a beautiful painting, or the composer a beautiful piece of music (according to their lights, at least). The mathematician, subject as he or she is to the imperatives of the two idols, truth and proof, tends to be surprised when something of genuine beauty emerges from his endeavours. Again, Brouwer would probably have dissented from this view, since he saw mathematics rather in the way that artists see art, namely as a free creative activity grounded in – or at least guided by – intuition, rather than a struggle to attain some absolute standard of truth.

7. *Beauty as richness flowing from simplicity*

The beauty of a mathematical concept often rests on the contrast between the simplicity and elegance of the concept itself and the richness and variety of the mathematical structures embodying it. This kind of mathematical beauty – let us term it *conceptual beauty*– in which variety emerges from simplicity – is analogous to the formal beauty found in musical works such as fugues (think of the mighty fugue in Bach's C major solo violin sonata – Bach's longest) – or sets of variations (such as Bach's *Goldberg Variations* or Beethoven's *Diabelli Variations*) in which the whole complex structure evolves from a simple germ.

Abstract algebra abounds in instances of conceptual beauty in this sense. The algebraic concepts of *group, ring, field, monoid, vector space, module, lattice* are simple and elegant and instances are encountered everywhere in mathematics.

Category theory provides one of the highest manifestations of conceptual beauty in mathematics. Resting on the underlying ideas of *transformation* and *composition,* its basic notions – category, functor, natural transformation, adjunction, topos – have a compelling simplicity and at the same time a vast generality which has enabled mathematicians to delineate the architecture of mathematics on a grand scale.

One of the most beautiful and fruitful definitions within category theory is that of (elementary) *topos.* A topos is a category which is Cartesian closed and has a subobject classifier. Cartesian closure is the condition – elegantly expressed in categorical terms – that, given any pair of objects *A, B* of the category there is an object

$A \times B$ which coordinates pairs of maps from arbitrary objects into *A* and *B;* and an object B^A which represents the ensemble of all maps from *A* to *B.* A subobject classifier is an object Ω which «represents» the parts of an arbitrary object *A* in the sense that there is a natural correlation between subobjects of *A* and maps $A \to \Omega$. A topos may be thought of as a mathematical «world» possessing its own internal logic (higher-order intuitionistic logic) there is a vast range of such «worlds» united by the topos concept, spanning the continuous and the discrete, the static and the varying.

A related type of conceptual beauty in mathematics is manifested by *axiom* (or postulate) *systems*, when a handful of simple postulates give rise to an extraordinary wealth of consequences. The classical example of an axiom system with such fecundity is of course Euclidean geometry. In Euclid's *Elements* 465 geometric propositions are derived from five simple postulates concerning points, lines and circles. Other examples include:

The axioms of set theory,

The axioms for the elementary theory of the category of sets.

The axiom of choice,

The Frege-Hume principle,

The Kock-Lawvere axiom of microaffineness, according to which curves are locally straight.

8. *Form and content in mathematics*

Form and content are at their closest in mathematics and music. In a musical composition the part of the «content» which is not embodied in the «form» is the emotional response it induces in the player or listener. Thus, professed musical formalists such as Stravinsky downplay the role in the composition of music played by the intent of triggering an emotional response in the listener. In mathematics the analogous thing to emotional response is the (recognition of) *truth.* So mathematical formalists deny that mathematical assertions can be «true» in the sense of correctly describing some objectively existing reality.

There have been a number of approaches to mathematics in which form and content are, in essence, *identified.* Here are some examples:

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Logicism in the foundations of mathematics, especially as propounded by Bertrand Russell (following Leibniz) , who attempted (unsuccessfully) to show that mathematics is ultimately reducible to logic, that the truth – the content – of mathematical propositions resides in their logical form. Mathematical truth is formal, logical truth.

The «propositions-as-types» doctrine in the constructive foundations of mathematics. Here a mathematical proposition is identified with a type of formal object – a *proof* or *demonstration.* The truth or content of a mathematical proposition then resides solely in form of its possible proofs.

The recently developed *Homotopy Type Theory* pivots on a bold identification of form and content in the so-called *univalence axiom.* In essence, this asserts that types or objects equivalent in form are literally *identical.*

Category theory. A mathematical category may be viewed as an explicit presentation of a *mathematical form or concept*. The category associated with a given mathematical form F has two constituents: *objects* and *maps.* Here objects are the embodiments of F, and maps are the transformations between the embodiments of F which in some specified sense «preserve» F. As examples we have:

Category	***Form***	***Transformations***
Sets	*Pure discreteness*	*Many-one correlations*
Sets with relations	*“ “ “ “*	*One-many correlations*
Groups	*Composition/inversion*	*Homomorphisms*
Topological spaces	*Continuity*	*Continuous maps*
Differentiable manifolds	*Smoothness*	*Smooth maps*

The subject matter of category theory is the form underlying, or uniting, all mathematical forms. It is (so far) the most complete realization of Paul Valéry’s claim that *mathematics studies the properties of a form, and no longer one particular problem.*

9. *Mathematics and fiction*

Despite Brouwer’s claim that mathematics is a «languageless activity», it seems evident that mathematics is language-based, both as a formal/symbolic practice and in its mode of transmission (through textbooks, lectures, etc.) It has been observed that mathematics resembles literary fiction in its systematic introduction of concepts such as numbers, circles, sets, etc. which are then *reified*, that is, treated as if they possessed independent existence—this is as true of constructive as of classical mathematics, by the way. In fic-

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tion, characters and events are treated, in accordance with Coleridge's «willing suspension of disbelief», as if they were real. Now one important difference between *classical* mathematics and the practice of fiction is that the reified concepts of the former, but not the latter, are treated as if their properties were fully determinate. For instance, it is accepted (I would surmise) by the majority of mathematicians that it is objectively determined whether the number $10^{10^{10}} + 3$ is prime or not – even if, as seems likely, *we* shall never know the answer. But in the case of fiction the case is otherwise. Scholars may debate Shakespeare's identity, but the question of whether Hamlet's breeches were, say, green, lacks determinacy, indeed borders on the absurd, since no scrutiny of Shakespeare's play could reveal their colour. Here the play is indeed the thing!

By contrast, the manner in which reified objects are treated both in constructive and in structuralist/axiomatic mathematics (category theory, for example) bears a closer resemblance to fiction. Constructive mathematicians acknowledge that the concepts and devices of mathematics are *invented* or *constructed*, even if under such (objective) constraints as to make it seem plausible later to describe them as the products of *discovery.* While in constructive mathematics *finite* objects such as individual natural numbers are treated as if their (finitistic) properties were fully determinate, (potentially) *infinite* objects such as the set of natural numbers, numerical functions, and individual real numbers are treated in a manner similar to fictional characters in that their properties are taken to be open to further determinations. The same can be said of structuralist mathematics. Just as Sherlock Holmes or Philip Marlowe have been the protagonists of a number of sequels to those works in which they made their debuts, so in structuralist mathematics there are a number of different ways of spelling out the properties of, for example, the real number system – «sequels», as it were, to its original conception. Models have been constructed in which every function on the real numbers is continuous, and also models in which every such function is computable. Like the practice of fiction, structuralist mathematics is *pluralistic.*

10. *Why is mathematical beauty so little appreciated by non-mathematicians?*

In conclusion, I offer a few reflections on the question: *why is mathematical beauty so little appreciated by non-mathematicians?* Mathematics is notoriously unpopular with, or at best misunderstood by, the general public. Indeed, many people have an aversion to mathematics. *Mathophobia* has been identified by educators as a serious malady afflicting high-school students. The very idea that there could be beauty in mathematics strikes the man-in-the-street as bizarre. The mathematician Paul Halmos

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when asked by someone what his profession was, reports that he was often tempted to reply, «I'm in roofing and siding». My own experience is similar. When someone asks me what my profession is, after first admitting to being a university professor, I hesitate to tell him what subject I profess. If I say «philosophy», his face lights up and he goes on to dilate enthusiastically on the meaning of life. On the other hand the confession that I am a mathematician (let alone a logician) is met with an embarrassed silence, broken only by my interlocutor's disconsolate admission «I was never very good at math».

I believe that the unpopularity of mathematics stems largely from the fact that the beauty of mathematics resides almost exclusively in the realm of the intelligible: it is thus regarded as «dry» and «difficult». In this regard music, which abounds in sensible beauty, offers a striking contrast. Musical activity involves three overlapping but essentially distinct classes of people: composers, performers and listeners. Composers and performers respond to both the intelligible and sensible beauties of music, while the vast majority of listeners respond solely to its sensible beauties. In (serious) mathematics, on the other hand, these three categories are collapsed into one. Yet mathematics is not *intrinsically* more difficult than music. at least in terms of the acquisition of technique. Mathematicians and musicians alike strive to develop and perfect their techniques. The difference is that the (performing) musician's struggle to perfect her art is finally crowned with popular success: no matter how difficult it may be to play a Beethoven sonata, a Paganini caprice, or to lay down an intricate jazz improvisation, an untutored ear can enjoy listening to, and respond to the beauty of the piece, despite lacking the slightest idea of how to play it. By contrast, the beauty of most mathematical creations is appreciated only by those mathematicians who are able, like performers of music, to recreate the mathematical work.

Mathematics would, I believe, become more popular if, in teaching the subject at the elementary level, greater efforts were made to convey its beauty – albeit of a purely intelligible kind – to students. To set the ball rolling, one might present examples of beautiful mathematical theorems whose meaning is readily grasped. These could include the Pythagorean Theorem, the theorem that there are exactly five regular polyhedra, Euler's polyhedron formula $V - E + F = 2$, Lagrange's theorem that every number is the sum of 4 squares, and the 4-colour theorem. Everyone is capable of appreciating the beauty of truths like these. All could then join Keats in drinking Newton's health, without further wishing, as did the poet, «confusion to mathematics».

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Bibliography

Chevalley, C., 1956: *Fundamental Concepts of Algebra*, Academic Press, New York.

Leibniz, G.W., 1714: *The principles of nature and of grace based on reason*, in *Philosophical Papers and Letters*, ed. Leroy E. Loemker, Reidel, Dordrecht 1970.

Leibniz, G.W., 1690-1698: *On Wisdom*, in *Philosophical Papers and Letters*, ed. Leroy E. Loemker, Reidel, Dordrecht 1970.

Leibniz, G.W., 1690-1698a: *On Felicity*, in *The Political Writings of Leibniz* ed. Patrick Riley, Cambridge University Press, Cambridge 1972.

Weyl, H., 1952: *Symmetry*, Princeton University Press, Princeton.

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