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# Mathematical beauty: On the aesthetic qualities of formal language

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**Abstract.** The paper proposes a reflection on mathematical beauty, considering the possibility of aesthetic qualities for formal language. Through a concise overview of the way this question is understood by some famous scientists and mathematicians, we turn our attention to Gian-Carlo Rota's theoretical proposal: his reflections as a mathematician and philosopher offer a perspective, of phenomenological matrix, fruitful for looking at the question. Rota's contribution allows us to focus on the role of competence, acquired through effort, sedimentation and habit of repetition, in cultivating the potential to recognise the aesthetic quality of formal language. Finally, we draw on some contributions from *Gestalt* theory, closely connected to twentieth-century phenomenology for chronological and conceptual reasons, applying the idea of *gestalt wholeness* to the mathematical product and proposing some reflections that may help to clarify some of the dynamics underlying the attribution of qualities of beauty to formulas or the detection of its lack.

**Keywords:** mathematics, beauty, phenomenology, Gestaltpsychologie, Rota.

> A physical law must possess
> mathematical beauty.
> Paul Dirac

Some time ago I was at the University of Milan, at the Department of Mathematics, on the occasion of a lecture I was about to give. Among the organisers were some long-standing friends, professors[1] in the field of Mathematics; in the minutes before the start of the lecture, they had started talking about the concept of "beauty". From my philosophical perspective, the discussion particularly

[1] One of them was Massimo Galuzzi, a professor of History of Mathematics and Computational Algebra, who passed away a few months ago. To him, a kind person, a humble scholar, always curious and open to dialogue with the human sciences, I dedicate this piece of writing as a sign of gratitude for what I learnt on all the occasions I was able to listen to him and dialogue with him.

struck me: the colleagues named some theorems in turn, which was followed by a unanimous and smiling acknowledgement for each one of how beautiful those formulas were. *They all spontaneously agreed*, for each equation mentioned, that it was something wonderful, even exciting. Later, at the end of the lecture, during the farewell greetings I came back to ask them a simple question, for which I was also a little embarrassed because I wasn't sure if it was an intelligent question: «when you talk about "beautiful theorems" are you all always in agreement?»

They replied that the agreement is usually extremely broad, almost unanimous: even if they have never worked together, if they do not know each other, even if they come from different parts of the world, mathematicians usually agree on whether a theorem is "beautiful" or not. Not only that: it is apparently common practice to intentionally seek out this feature, pursue it, and highlight it if present. As I discovered shortly afterwards, the issue was more extensive than I naively thought. «Mathematicians enjoy discussions of the beauty of mathematics» (Rota [1997]: 121); although "beautiful" is a problematic term, evidently far removed from the rigour of the exact sciences, members of this scientific community «are fond of passing judgment on the beauty of their favoured pieces of mathematics» (Rota [1997]: 121). What impressed me immediately was what seemed to be a singular use of this aesthetic judgement: it is a case, I do not know if unique but certainly rare, in which it seems very difficult to pass judgement if one is not really an expert in the discipline.

When confronted with a painting, a symphony, a monument, it is not indispensable to know how to paint, compose, sculpt, draw or realise an architectural project in order to express an aesthetic opinion. The latter may not be decisive for the advancement of studies, it may certainly be a naïve opinion if the person formulating it has little or no skill or expertise, but it remains possible to express an opinion on it and possibly enjoy it. When faced with a mathematical formula, a different situation arises: what supposed aesthetic aspects could I possibly evaluate in a string of formal language if I do not know its meaning? Some objection might arise at this point: it could be a question of two different categories of "beauty", or a misuse of the term in the case of mathematics. Why is it that even I, who do not know how to use oil colours, recognise the beauty of a Monet but have no element that allows me to "feel" that of Euler's identity?

The paper first proposes a reconnaissance of the reflections on mathematical beauty proposed by a number of famous scientists, in order to offer a concise overview of the way this question is understood by those who have practised mathematics professionally at the most expert levels. We then turn our attention to Gian-Carlo Rota's theoretical proposal: his reflections as a mathematician and philosopher offer a perspective, of phenomenological matrix, that we consider fruitful for looking at the question. In particular, Rota's contribution allows us to focus on the role of competence, acquired through effort, sedimentation and habit of repetition, in cultivating the potential to recognise the aesthetic quality of formal language[2]. Finally, we draw on some contributions from *Gestalt* theory, closely connected to twentieth-century phenomenology for chronological and conceptual reasons, applying the idea of *gestalt wholeness* to the mathematical product and proposing some reflections that may help to clarify some of the dynamics underlying the attribution of qualities of beauty to formulas or the detection of its lack.

## BEAUTY AS A CRITERION FOR SCIENTIFIC RESEARCH

The sentence at the beginning of this article was written, chalk on slate, on a blackboard at Moscow University: Paul Adrien Maurice Dirac,

[2] In this investigation, therefore, we are particularly concerned with formal mathematical language, although we do not exclude the possibility of extending our discourse to logical language; indeed, we are inclined to assume that the argumentation would work in the same way, but we reserve the right to conduct further studies.

Nobel Prize in physics, was there to give lectures in 1956. He wrote this statement as an answer to the question put to him by some of the listeners to summarise the principles of his own philosophy of physics. A great expert in mathematics, Dirac held the principle of mathematical beauty in the highest regard, using it both as a «heuristic guide» and, less commonly in his own scientific community, even as an «evaluative criterion» (Barone [2019]: 18-19; McAllister [1990]). The search for the beautiful formula or demonstration – which for some coincides with an evaluation of "elegance", but it would seem that this does not apply unanimously (Rota [1997]: 128) – is a well-established and recognised practice in mathematics; it is perhaps less so in physics, as the need for experimental verification of results usually acts as a counterbalance to mathematical work alone and the intention to craft a perfect formula.

However, Dirac firmly believed that research in physics should be directed by the choice of mathematics to be used as the basis for one's theory, and that mathematics should, first and foremost, be *beautiful*. The scientist believed that the «most powerful method of advance» that could be suggested for theoretical physics was to «employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and *after* each success in this direction, to try to interpret the new mathematical features in terms of physical entities» (Dirac [1931]: 60).

According to him, working on an aesthetically satisfying mathematical product would be a good guide to good results in physics; in fact, from his own experience he concluded that «it seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power, needing quite a high standard of mathematics for one to understand it» (Dirac [1963]: 247).

We can also say that the proposal to start from mathematics, in physics, is an authentic «methodological revolution» (Barone [2019]: 27) introduced by Dirac; in this way mathematical work, and specifically the search for a pleasing form of it, would assume a hierarchically superior position to experimental confirmation. Far more singular is his firm belief in an evaluative function of the mathematical part, which could even impose itself on the experimental one; indeed, Dirac was convinced that, in the presence of beautiful mathematics and «really a sound insight», then the physicist could be confident of being «on a sure line of progress» (Dirac [1963]: 241). But what to do in the event of no experimental confirmation? A famous example he used to mention was an affair involving Erwin Schrödinger, who in the 1920s was working simultaneously and independently on the same problem that Werner Heisenberg was also working on, namely the emerging quantum theory.

The former, moving from an eminently theoretical perspective in contrast to the latter and searching for «a beautiful theory» (Dirac [1963]: 240), arrived at a wave equation describing atomic processes that had great beauty. However, when put to the test, it seemed to contradict the results of experiments. We now know that the equation was valid, and was indeed better than the one Schrödinger ended up publishing in its place: a non-relativistic version that, as Dirac points out, had inferior aesthetic qualities, obtained to make the experiments fit. In fact, what prevented the experimental verification from being correctly understood was the fact that the spin of the electron had not yet been discovered at the time. Dirac used this example to emphasise that Schrödinger should have had faith in the beauty of his equation, and that «it is more important to have beauty in one's equations than to have them fit experiment» (Dirac [1963]: 241).

Indeed, «if there is not complete agreement between the results of one's work and experiment, one should not allow oneself to be too discouraged, because the discrepancy may well be due to minor features that are not properly taken into account and that will get cleared up with further developments of the theory» (Dirac [1963]: 241). Nor was Dirac totally alone in this position; just to give an example, the search for precise aesthetic qualities in one's mathematics was also dear to the

German physicist Hermann Weyl, whose confession was: «my work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful» (Chandrasekhar [1987]: 65).

But what exactly is meant by "mathematical beauty"? Dirac is lapidary on the subject, and does not, unfortunately, offer much in the way of clarification. In fact, the scientist claims that «this is a quality which cannot be defined, any more than beauty in art can be defined, but which people who study mathematics usually have no difficulty in appreciating» (Dirac [1939]: 122). He adds an element, stating that «it often happens» that «the requirements of simplicity and of beauty are the same» (Dirac [1939]: 122); in this regard, it is necessary to understand the meaning of the adjective "simple". It would not be «practical simplicity» as «ease of description or calculation», but rather «logical simplicity, understood as conceptual economy» (Barone [2019]: 20).

We can say, therefore, that according to Dirac, the beautiful equation or demonstration usually bears a small number of elements, is organised according to an uncomplicated structure, and consists of a small number of primary concepts. It is also evident that this cannot be an idea of trivial ease of comprehension or calculation: consider that what Dirac considers most beautiful in theoretical physics include formulae such as Einstein's law of gravity and his own wave equation, which describes the motion and properties of electrons in a relativistically invariant manner; these are evidently examples of extremely sophisticated mathematics.

Other mathematicians have tried to define this concept which, as in any other field, continues to elude strict definition: as Godfrey H. Hardy, a British mathematician also famous for being the mentor of the eminent Indian mathematician Srinivasa Ramanujan, recalls, «it may be very hard to define mathematical beauty, but that is just as true of beauty of any kind» (Hardy [1940]: 14). Attempting to clarify its meaning, Hardy links it to a property he calls «seriousness»: a mathematical idea would be serious, in this case, if characterised by «a certain *generality* and a certain *depth*» (Hardy [1940]: 15).

We are again in the presence of other concepts that are difficult to define, and Hardy is aware of this, although he does attempt to sketch something useful about them. The "generality" would have to do with a wide extension or a vast network of relations with other mathematical constructions; we can think of a kind of evaluation of the number of questions with which the theorem in question would be connected. As he points out, for instance, there are many valid demonstrations of properties of groups composed of a few numbers, but these are mathematical curiosities, of which one would not say that they are "serious" because they lack sufficient "generality"; they have no chance of being beautiful (Hardy [1940]: 24-25).

Generality, defined somewhat vaguely by Hardy, would also have something to do with "depth": if a theorem is placed in connection with numerous other mathematical constructions, it will therefore constitute an important node in the network that constitutes this discipline. If, instead of imagining a network, we think of a stratification of discoveries and ideas, we can say that the "deepest" ones would lie at the bottom of the others, not as the last in a hierarchical sense, but in the function of basis and support (Hardy [1940]: 27-28). So far, these are considerations that go beyond looking at the individual formula, focusing rather on the position of the mathematical product in the broader scientific landscape of the discipline.

This could also be considered in terms of the competence required to understand the theorem, or the proof under consideration: "serious" mathematical work, in Hardy's sense, would require a considerable background of knowledge to be appreciated. The mathematician also identifies other qualities that would be present in beautiful mathematics, of which he brings Euclid's and Pythagoras' theorems as examples: «unexpectedness», «inevitability» and «economy», which Hardy calls «"purely aesthetic" qualities» (Hardy [1940]: 29). These are, once again, terms that are not easy to define if one wants to be rigorous.

The author sketches a few lines of clarification, saying that in these theorems «the arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions» (Hardy [1940]: 29). So: the beautiful formula would have something unexpected in its configuration, which is emphasised as a matter of observation; there returns, as in Dirac, a valorisation of simplicity, in the sense of cleanliness and conceptual economy; finally, a reflection is added on a sort of *necessity* effect of the result obtained. We might consider that in any well-crafted logical deductive chain, the consequence necessarily follows from the premises, but this would apply to every theorem proved. Instead, in this case, there is a feeling of conclusiveness in the face of a form or structure that, as per the classical definition of the adjective "necessary", *cannot be otherwise*.

Let us also add to this brief overview the position of Heisenberg, who, when confronted with our topic, proposes a suggestive consideration that helps to better understand this theme of logical simplicity that has already emerged in previous cases. The scientist correlates beauty with the idea of unity in multiplicity: «the fact that in such a theory the many are confronted with the one, that in it the many are unified, itself has the undoubted consequence that we also feel it at the same time to be simple and beautiful» (Heisenberg [1974]: 174); he also mentions, in this regard, a characteristic of «proper conformity of the parts to one another and to the whole» (Heisenberg [1974]: 174). We believe that exploiting the concept of unity as a balanced and harmonious configuration of several elements may help to clarify part of this problematic issue. We propose to make use of Gian-Carlo Rota's contribution at this point, as a mathematician and philosopher whose theoretical roots lie in Husserlian phenomenology: we are thinking of the *Logical Investigations* and especially the third one, which concerns reflection on the whole and the parts.

Without delving into this genealogy for obvious reasons of economy of space, but only with the intention of fixing a useful point in the network we are trying to build, we can now propose a few themes elaborated by Rota.

## EFFORT AND STEPWISENESS

In an essay dedicated to the *Phenomenology of Mathematical Beauty*, Rota acknowledges that the term "beauty" is widely used by the scientific community to which he also belongs, so it is a widespread and important phenomenon to subject to analysis. The author believes that there is a rather neglected element in the common, and nonetheless valid, practice of leveraging the heuristic function of beauty in mathematics. Indeed, he believes that, similar to «creativity» and even «happiness», beauty is a desirable but not directly pursuable result: it would not be possible, in his opinion, to obtain a mathematical product with positive aesthetic qualities on command. These would result as an «unpredictable byproduct» or «benefit», a consequence of mathematical activity that can arise, certainly, but by surprise, without the possibility of planning a strategy that leads directly to the result (Rota [1997]: 127). With regard to the ability to appreciate beauty in a mathematical work, which as we have already considered seems to be reserved for those who at least adequately understand its meaning, Rota believes that it is not a teachable or transmittable ability as one does with knowledge. According to the author, a mathematician comes to appreciate beauty in his field when he achieves a consistent «familiarity» with mathematical theory. This concept, for which Rota draws on Heideggerian vocabulary, serves to describe the «different degrees» of «our involvement» (Rota [2019], Part VI, Chap. 2; Rota [1997]: 128) in a given context and is linked to Rota's broader reflection on teaching and learning. In order to become familiar with a theory one needs a consistent and solid habit of work and exercise: one needs «stepwiseness», «effort» and «*Sitzfleisch*». The latter German term is a combination of «sits» and «flesh», which Rota uses to designate «the ability to sit at a desk for ten hours in a row without getting up»; by getting used to keeping one's body physically still in front of the problem, and concentrating in a prolonged and constant manner, exercise becomes habit, hence sedimented knowledge that leads to feeling "familiar" with the subject matter.

We would add that in other fields too, a certain habit and sedimentation of knowledge is indispensable, which we can also think of in terms of prolonged exposure to fashions and aesthetic trends, in order to judge the beauty of a product; but these could probably be skills that require a less intense and all-encompassing intellectual effort than that required in the mathematical field. We limit to providing this suggestion, since it does not deal with the specific subject of this paper, but considering it useful to propose this comparison.

Thus, for Rota, the laborious and constant sedimentation of learning is indispensable to the perception of the so-called "beauty" of a theory. This may seem strange, since, as he himself notes, «we think back to instances of appreciation of mathematical beauty as if they had been perceived in a moment of bliss, in a sudden flash like a light bulb suddenly being lit». Indeed, it happens that the familiarity acquired allows us to remove the «painful process of learning», leaving in its place the illusion of a «flash of insight» (Rota [1997]: 130). A mathematician therefore has an experience that, like perceptual experiences, does not seem to carry a burden of reasoning or presuppose habitual work, but appears instantaneous, easy, immediate. Rota devises a name for this misunderstanding: «light bulb mistake»; it would consist in the rather common illusion whereby mathematical truth would be grasped in an instantaneous transition from darkness to light, like a dazzling intuition. This view, which also often conditions the success of mathematics teaching in that it leads students and teachers to imagine that some kind of miraculous illumination is needed to reach understanding, neglects all the effort, habit and exercise that are indispensable to understanding meaning: «appreciation of mathematical beauty requires familiarity with a mathematical theory, and such familiarity is arrived at the cost of time, effort, exercise, and *Sitzfleisch*» (Rota [1997]: 128).

In constructing an analysis of the phenomenon of mathematical beauty, Rota proceeds by looking at instances of theorems or demonstrations that mathematicians consider lacking this characteristic. The author notes that they do not usually tend to use the term «ugly», but rather other terms such as «awkward», «obscure», «redundant», «pointless»; furthermore, a recurring reaction is produced, which he defines as a «rhetorical question» and which he summarises as follows: «what is this good for?» (Rota [1997]: 129).

Now, even the layman can understand that pure mathematical research, before it finds an application, is not something immediately useful; it can subsequently, through encounters and intersections that are also fortuitous, intercept various needs of other applied sciences such as engineering, statistics, economics – just to give a few examples – and only then can it become useful. «Most results in pure mathematics, even the deepest ones, are not "good" for anything», Rota points out; yet this question recurs when faced with bad mathematics. Rota provides an explanation related to the *sense*: he believes that this supposed "usefulness", which mathematicians ask about, concerns not so much the practical use of the formula as its ability to "enlighten" the reader as to its meaning.

To follow up on this brief and difficult argument, we think it may be useful to recall from Rota's philosophical work the idea of *Fundierung*, which he takes from Edmund Husserl's *Logical Investigations* and uses to reason about the question, also Husserlian, of the relationship between the whole and the parts. The term *Fundierung* indicates a structural foundation relation that is established between the material aspect, called *facticity*, and *function*, which Rota also often calls *sense* (see Rota [2019], Part I, Chap. 5 ). The example of chess is useful to understand the difference and the link between the two aspects: «the game of chess requires pieces to be played. Nonetheless, we cannot infer anything about the game of chess by staring at the pieces» (Rota [2019], Part I, Chap. 3). In this case, the materiality of the chess pieces, the material they are made of, their shape and size, constitutes the factual aspect; the role of each piece and the moves allowed to it, the set of rules of the game that is conducted with those pieces, constitutes the function or «sense». The latter is «layered upon» the facticity, although it is not reducible to it; the «phenomenon whereby a facticity is transcended towards sense» is described by Rota using the Heideggerian concept of *Ereignis*.

Now, since «you sense through a facticity», although the former is not reduced to the latter, the latter is an indispensable condition of the former, and facticity also partially determines our possibility of grasping meaning. If we think of a theorem, a formula, its facticity will be constituted by the signs that compose it, by the chalk on the blackboard or the ink on the paper, by the signifiers we hear or think of. Since these must have been chosen from among others, and placed in some configuration, we believe that the order in which those very symbols are placed and organised can go to constitute a *form*, a whole, which according to its organisation can be a better or worse vehicle of meaning: more or less *enlightening*.

For Rota, «mathematical beauty is the expression mathematicians have invented in order to obliquely admit the phenomenon of enlightenment while avoiding acknowledgment of the fuzziness of this phenomenon. They say that a theorem is beautiful when they mean to say that the theorem is enlightening»: this is therefore a kind of misappropriation, a misuse of the term[3].

First of all, let us make it clear that we are not discussing the "truth" of which mathematical work should be the bearer, since any rigorously demonstrated work, whether beautiful or ugly, will certainly be true in the mathematical sense. Instead, it is a question of the ability that a string of formal language, given its construction and form, would or would not have, to make us grasp «the *sense* of the statement that has been verified to be true» (Rota [1997]: 131), an effect that Rota calls «enlightenment». We are again faced with a term that is proposed to explain a very vague one, i.e. "beauty", but which is equally difficult to define; Rota himself calls this matter "fuzzy". How to recognise an enlightening theorem? It is if it «"fits" in its place», when «it sheds light around itself»; above all, an observation we consider somewhat clearer, when it is structured in such a way as to lead us to «perceive the inevitability of the statement being proved» (Rota [1997]: 132).

Therefore, here too appears one of the terms that had already been identified by other thinkers to define our problem: inevitability, which we can think of as the necessity of arriving at precisely that destiny, that configuration. Rota also believes that «lack of beauty is related to lack of definitiveness», and very helpfully points out that «a beautiful proof is more often than not the definitive proof (though a definitive proof need not be beautiful); a beautiful theorem is not likely to be improved upon, though often it is a motive for the development of definitive theories in which it may be ensconced» (Rota [1997]: 128-129). This is why, when there is a lack of beauty in a mathematical work, then «a motivation for further research» arises; indeed, most mathematical research work certainly does not consist of purely original results, which are quite rare, but rather «of polishing and refining statements and proofs of known results»: improving a form which is already true and rigorous, but which lacks definitiveness. «Beauty is seldom associated with pioneering work», the mathematician continues, recalling that «the first proof of a difficult theorem is seldom beautiful» (Rota [1997]: 129), and when one finds the beautiful form one is generally ready to consider the matter "closed" and move the search elsewhere.

## A GOOD SHAPE

To appreciate mathematical beauty, therefore, requires not only basic but advanced competence; habit, exercise, *Sitzfleisch*. As Rota recalls, «the fundamental theorem of calculus depends for its

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[3] For a different perspective on the subject from the one proposed here, see Cellucci [2015] and, more widely, Montano [2013].

sense on knowing a certain amount of mathematics. Take away the math and it becomes incomprehensible» (Rota [2019], Part I, Chap. 5), the sense fades out, and with it any possibility of assessing whether it is nice or ugly.

The doubt arises that there may be some purely aesthetic qualities in mathematical formulae: if one needs all this knowledge and preparation to enjoy them, are we not detaching ourselves from the spectrum – which is also wide – of the aesthetic phenomenon, in its connection with the perceptive?

A non-specialist audience, but one endowed with an amateur curiosity for mathematics, may indeed also express themselves on the beauty of the formulae [4]; for example, some characteristics that may positively strike the aesthetic sense of those outside the discipline are: the brevity of the formula, the presence of symbols whose meaning they know, a certain "symmetry" of the writing between the first and second part of the equation[5].

However, «theories that mathematicians consider to be beautiful seldom agree with the mathematics thought to be beautiful by the educated public» (Rota [1997]: 122); Rota brings up the example of Euclidean geometry, pointing out that, if this arouses a fairly widespread appreciation on the part of people who are generally educated but not experts in the field, we cannot really say that for professional mathematicians it arouses the same enthusiasm. Thus, from what has been said so far we can deduce that the "outward appearance", so to speak, of the string of formal language is not the exclusive seat of its eventual beauty; however, it does not seem to be possible to exclude this aspect entirely from the question. This is because: 1) the configuration of the formula is a conceptual matter but also a "visual" one; 2) adopting Rota's perspective, we arrive at meaning through facticity, and this as a material aspect implies that it can still be the object of perception at the moment the formula is read, seen, heard.

It is certain that beauty for mathematicians is a serious matter: «a piece of mathematics that is agreed to be beautiful is more likely to be included in school curricula; the discoverer of a beautiful theorem is rewarded by promotions and awards; a beautiful argument will be imitated. In other words, the beauty of a piece of mathematics does not consist merely of the subjective feelings experienced by an observer» (Rota [1997]: 126). Thus, if a scientific community accustomed to rigour, which makes use of a symbolic language cleansed of all vagueness and ambiguity compared to the natural one, unanimously uses precisely the category of "beautiful", which has always been problematic to define with rigour, there must be a reason. Therefore, recognising and appreciating Rota's decisive contribution to this issue, which we have reported precisely because we believe in its usefulness, we note that in our view perhaps it is not very appropriate to think of a misuse of the word "beauty" by mathematicians, who according to the author, as we have seen, would choose it to avoid speaking of "enlightenment". Also considering that such a substitution would not seem to improve things from the point of view of conceptual clarity, we think it more likely that if mathematicians use the concept of beauty, and precisely this word and not another, it is because it is precisely a type of beauty like any other.

It does not, in our view, differ qualitatively, but as the amount of habit and sedimented effort required – necessary but not sufficient – to be able to enjoy and grasp it, which are particularly high in the case of mathematics; we do not know and do not wish to express ourselves on other types of beauty as this is outside the context of this con-

[4] In this regard, see the interesting study by Zeki et al. [2014], from the neuroscientific field, in which we can see the radical difference in response between mathematicians and non-mathematicians exposed to the same stimuli and asked questions about the beauty of certain equations and the possible emotional response that arises in them upon reading.

[5] See Eugeni, Nicotra [2019]: 91-92. What is meant here is precisely the "drawing" sketched by the signs constituting the formula, thus a consideration «like observing an Escher drawing», which therefore 1) is unrelated to the meaning of the formula, 2) has nothing to do with the specific sense that the term "symmetry" takes on in the sciences, especially in mathematics used for physical laws. See on this subject Barone [2013].

tribution and would require a separate study (see Portera [2021]).

If the only characteristic needed to evaluate the aesthetic quality of a theorem were competence, it would be difficult to think of a distinction between beautiful and unbeautiful formulas, since every mathematical work needs competence to be treated. Therefore there must be some other aspect; as we have seen, illustrious thinkers have attempted to evoke this with terms such as *unity*, *inevitability*, *simplicity*. Picking up and exploiting the references proposed so far to forms and configurations of multiple elements in a whole, we would like to note that these attributes can be included in the characteristics that, within the framework of *Gestalt* theory, are attributed to what is called a «good form»: «regularity, symmetry, cohesion, homogeneity, balance, maximum simplicity, conciseness» (Katz [1992]: 60).

Through Rota, in contact with Husserlian phenomenology, at this point availing ourselves of *Gestalt* theory does not seem unsuitable but appropriate; for these two fields show extensive intersections, so that one can be said to owe much to the other and vice versa.

A possible objection immediately arises: what *form* should be considered, in the case of an equation for example, if we have so far said that it is eminently the non-specialist public that would take into account the design traced by symbols? To answer this, we can appeal to the studies of the psychology of form in the field of «theory of thought». Indeed, in this field of study it is possible to speak of form not only for perceptual wholes, such as visible configurations (figures) or audible ones (melodies), but also for wholes that emerge in other types of noetic interaction – to put it in a phenomenological lexicon. Indeed

> I can hear a very long sentence – read or heard – as a unity. A dialogue conducted over a fairly considerable period of time can be in my consciousness as a unity. The discussion of a problem by a scientific association can be conceived by the various members as an integral unity. How could I understand the laborious demonstration of a mathematical axiom, if it did not present itself to my consciousness as an integral whole, in such a way that the individual elements of the demonstration remain alive, well interlocked with each other? (Katz [1992]: 57)

In such cases, we can speak of «cogitative forms or structures», which, although they differ from figural forms, also have certain properties in common with them and constitute, as much as these, «unitary complexes». For Kurt Koffka, one of the leading exponents of the psychology of form, thought itself as a structured process would function through laws analogous to those that apply to sensation.

This theoretical perspective would, in our opinion, allow us to hypothesise a possible interpretation of mathematical beauty in continuity with Rota's one, which could perhaps – we certainly do not want to say *resolve* the issue, but – help shed some light on it or at least provide useful elements for reflection.

Let us consider, among the Gestalt laws, the so-called "law of Prägnanz"; in recalling it, let us remember that the Gestalt school does not propose a single definition for each law but, in an agreement of principle among the various theorists, brings some differences in the enunciation of the basic principles. The Prägnanz manifests itself through the so-called «good form» concept, of which we offer the following definition: «psychological organisation is always as "good" as the given conditions allow» (Katz [1992]: 60; see also Koffka [1935]).

This entails another necessary step: in cases where the form is not good, i.e. if we are faced with an «"open" or incomplete figure», for *Gestalt* psychology we inevitably tend to want to complete and "close" that form in order to achieve a stable and satisfactory, *definitive* configuration. Even in the sphere of thinking activity, and not only in the face of visual or auditory configurations, there is the need to bring the configuration to a stable point of equilibrium: «there is an analogy between the tendency to Prägnanz as, for example, manifested in the phenomenon of the integration or "closure" of an "open" or incomplete figure, and the integration or closure of an open question, considered as a "cogitative form"» (Katz [1992]: 120).

We note that the attributes of the good cogitative form are similar to the attributes of the beautiful equation: a stable, closed and therefore definitive whole; in equilibrium, whose constituent elements are simple in the sense of logical optimisation and reduction to the least possible complexity; placed in such an order that no element needs to be moved any further, giving a sense of *inevitability*.

On the contrary, a bad or open form carries a tension in the observer that translates into the need for closure, rearrangement, repositioning of the elements until a stable form is found. This is reminiscent of what Rota said about everyday mathematical research, which proceeds through efforts to refine and "embellish" matters that are already rigorous and true but not yet final: these do not possess mathematical beauty and therefore one strives, while still having something that works for what it must, to arrive at a stable and closed form that allows one to say "we can stop".

We can continue the analysis thus far by briefly considering the contribution of the psychology of form to the study of productive thought, that is, the activity of thought that creates and produces knowledge. According to Max Wertheimer, this activity essentially consists of «structural transformations», during which «facts must undergo a polarisation around new centres» (Katz [1992]: 122; see Wertheimer [1959]). It would be a matter of using the same facts, i.e. elements with which one is working, and reorganising them in a new configuration in which the equilibrium point can change, in order to achieve greater stability.

A particularly fruitful example, although it does not concern mathematical beauty but nevertheless allows us to better explain the question, is given by the gestalt reading of some famous "serendipities" in the history of science, such as the falling apple for Newton or the swinging chandelier for Galileo. Obviously, Newton was not the first individual to see an apple fall, nor can we think that anyone saw a chandelier swing before Galileo; the question therefore arises as to why such events could constitute such powerful inspirations. One possible interpretation, certainly not the only one but in our opinion a convincing one, is the gestalt perspective of Prägnanz. The great scientists mentioned were, in those moments, in a kind of «psychic tension» because of the open and urgent questions that were grinding within them, waiting impatiently for closure. We can imagine the issues that were churning within them as large constructions, already sufficiently mature but not yet closed, generating a «tension» that «required balance». Then «their casual observation of the apple or the chandelier came to fill their mental "gap", just as the last piece of a puzzle fits the last remaining gap in the game» (Katz [1992]: 122-123).

Thus, the idea of considering mathematical expression as a gestalt whole, and thinking of cases of ugliness as open forms in need of reconfiguration and cases of beauty as closed and definitive forms, seems to us an interesting possibility for reflecting on this complex issue.

Moreover, appealing to a theory that begins by working on sensible forms and goes on to extend its concepts to "cogitative" configurations allows us to reconcile the conceptual aspect, preponderant in the experience of formal mathematical language, with the perceptual one, which we also believe is present. A formal string in fact remains in any case a linguistic expression, however non-natural, and as such it does not seem strange to attribute aesthetic qualities to it.

In this regard, Maurice Merleau-Ponty's reflections on «algorithmic language», the definition he gives to the formal language of logic and mathematics, may be useful, again drawing on the field between phenomenology and the theory of form. Although it has been cleansed of the ambiguity and historicity that characterise it in its natural version, it is still language: «algorithmic expression […] is a special case of language» (Merleau-Ponty [1973]: 128).

But just as knowing how to read is not enough to appreciate the beauty of a novel, neither is knowing the individual meanings of the symbols used in a formula enough to assess its aesthetic impact: it is a necessary but not sufficient condition. Indeed, «mathematical truth» and with it eventual beauty, we add, «appears only to a subject for whom there are structures, situations and a perspective», i.e. a competence, just as «"language" knowledge arouses transformations in the given significations which were contained in it only in the way French literature is contained in the French language» (Merleau-Ponty [1973]: 131); that is to say, competence, exercise, stratified habit are also needed to be able to appreciate a literary work. Let us add, moreover, that if we can consider the hypothesis that it is not ambiguity, vagueness and autonomous diachronic evolution that allow for the aesthetic qualities of languages, but that these may be based on the symbolic function as such, then we may have some more elements to understand why a formal string can move competent people.

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